CHAPTER XIX. Lagrangian Modelling for Turbulent Flows

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Turbulence models for calculating the properties of inhomogeneous turbulent flows are generally based on Eulerian statistics[1-3]. Lagrangian methods have a long history in the study of turbulence[4], but have yet to be fully exploited. We briefly review here the use of a Lagrangian model - namely the generalized Langevin equation. Obukhov[5] showed that the simple Langevin equation is consistent with Richardson and Kolomogorov inertial-range scaling laws, and this model has subsequently formed the basis of many dispersion calculations[6]. The generalization of the model to anisotropic turbulence with mean velocity gradients is discussed, and it is shown that (with a knowledge of the dissipation rate ε) the generalized model provides a closure of the transport equation for the one-point Eulerian velocity joint probability density function.

Let $\underline{x}^+(\underline{\xi},t)$ and $\underline{U}^+(\underline{\xi},t)$ denote the position and velocity of the fluid particle that is located at $\underline{x}=\underline{\xi}$ at a reference time t_o. In the infinitesimal time interval dt, the fluid particle (by definition) moves by

$$d\underline{x}^+ = \underline{U}^+ dt \ . \qquad [1]$$

According to the generalized Langevin equation, the corresponding velocity change is

$$dU_i^+ = -\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_i}\,dt + G_{ij}(U_j^+ - \langle U_j\rangle)dt + \sqrt{C_o\varepsilon}\;dW_i ; \qquad [2]$$

where ρ is the density (assumed constant); $\partial\langle p\rangle/\partial x_i$ and $\langle U_j\rangle$ are the mean pressure gradient and mean Eulerian velocity at the fluid particle location $\underline{x}^+$; C_o is a constant; G_{ij} is a second-order tensor function of local mean Eulerian quantities; and $\underline{W}$ is an isotropic Weiner process, so that $d\underline{W}$ is a joint normal random vector with zero mean and covariance

$$\langle dW_i dW_j\rangle = dt\ \delta_{ij} \ . \qquad [3]$$

Haworth and Pope[7] describe a model for G_{ij} in terms of the local mean velocity gradients, the Reynolds stresses $\langle u_i u_j \rangle$, and the dissipation rate ε. The time scales associated with G_{ij} are then the time scales of the mean deformation and the dissipation time scale $\tau \equiv 1/2 \langle u_i u_i \rangle / \varepsilon$.

For time intervals s that are much smaller than τ, the random term in Eq. (2) is dominant, and the Lagrangian structure function is (to first order in s)

$$\langle [U_i^+(t+s) - U_i^+(t)][U_j^+(t+s) - U_j^+(t)] \rangle = C_o \varepsilon s \, \delta_{ij} \; . \qquad [4]$$

As first observed by Obukhov[5], this shows that the Langevin equation is consistent with Kolmogorov's inertial-range scaling laws. Further, Eq. (4) identifies C_o as a universal Kolmogorov constant.

For isotropic turbulence in the absence of mean velocity gradients, the tensor G_{ij} is (without further assumption)

$$G_{ij} = -\delta_{ij} \left(\frac{1}{2} + \frac{3}{4} C_o\right) / \tau \; . \qquad [5]$$

Anand and Pope[8] used this Langevin equation (Eqs. 2 and 5) to calculate the dispersion of heat from a line source in grid turbulence. (The term $\sqrt{2\alpha}\, d\underline{W}'$ was added to Eq. 1 to account for thermal conduction; α is the specific diffusivity and $\underline{W}'$ is a second Weiner process.) Figure 1 shows the calculated thermal wake thickness compared to the available experimental data. On the basis of this comparison, the value 2.1 was determined for the Kolmogorov constant C_o.

For anisotropic turbulence with mean velocity gradients, Haworth and Pope[7] proposed a model for G_{ij} that is linear in $\langle u_k u_\ell \rangle$ and $\partial \langle U_m \rangle / \partial x_n$. This model (which contains four constants) is capable of reproducing the measured evolution of the Reynolds stresses in all homogeneous flows for which there are data. As an example, Fig. 2 shows the evolution of the anisotropies

$$b_{ij} \equiv \langle u_i u_j \rangle / \langle u_\ell u_\ell \rangle - \frac{1}{3} \delta_{ij} \; , \qquad [6]$$

for one of the plane strain experiments of Gence and Mathieu[11]. This is a particularly testing case in that, initially, the principal axes of the Reynolds stress tensor are at 45° to those of the mean rate-of-strain tensor.

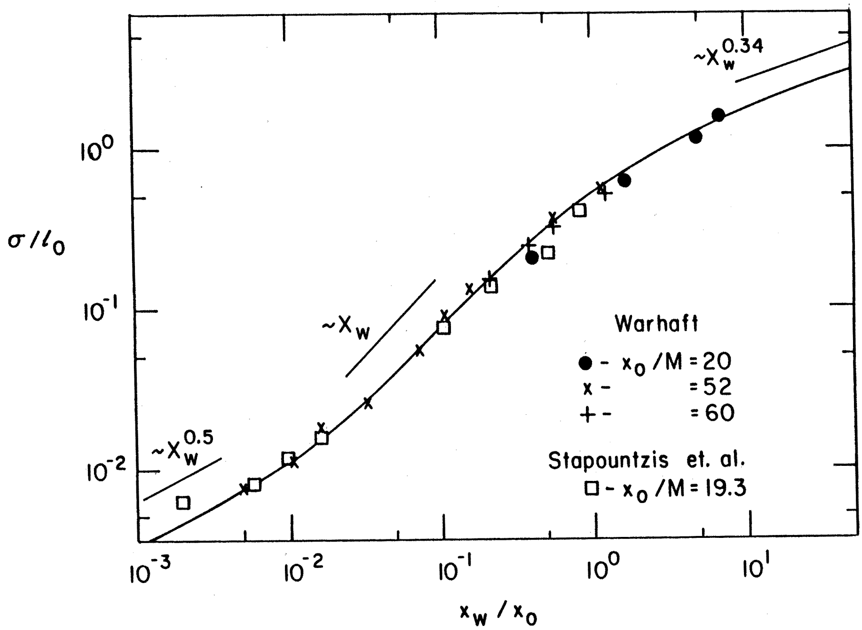

Fig. 1: Thermal wake thickness σ normalized by the integral scale at the heated wire ℓ_0 against distance from the wire x_w normalized by the distance from the grid to the wire x_0. Langevin equation calculations (line) Anand and Pope[8]: experimental data (symbols) Warhaft[9] and Stapountzis et al.[10]

For inhomogeneous flows, the significance of the Langevin equation can be understood in terms of the Eulerian velocity joint probability density function (pdf). With $\underline{U}$ being the Eulerian velocity, $f(\underline{V};\underline{x},t)$ is defined to be the joint probability density of the event $\underline{U}(\underline{x},t)=\underline{V}$. According to the Navier-Stokes equations, the joint pdf evolves[12] by

$$\frac{\partial f}{\partial t} + V_i \frac{\partial f}{\partial x_i} - \frac{1}{\rho}\frac{\partial \langle p\rangle}{\partial x_i}\frac{\partial f}{\partial V_i} = -\frac{\partial}{\partial V_i}\left[f\langle -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu\nabla^2 U_i \big| \underline{V}\rangle\right] , \qquad [7]$$

where p' is the pressure fluctuation and ν is the kinematic viscosity. On the other hand, from the Langevin model we obtain

$$\frac{\partial f}{\partial t} + V_i \frac{\partial f}{\partial x_i} - \frac{1}{\rho}\frac{\partial \langle p\rangle}{\partial x_i}\frac{\partial f}{\partial V_i} =$$

$$- G_{ij}\frac{\partial}{\partial V_i}\left[f(V_j - \langle U_j\rangle)\right] + \frac{1}{2} C_0 \varepsilon \frac{\partial^2 f}{\partial V_i \partial V_i} . \qquad [8]$$

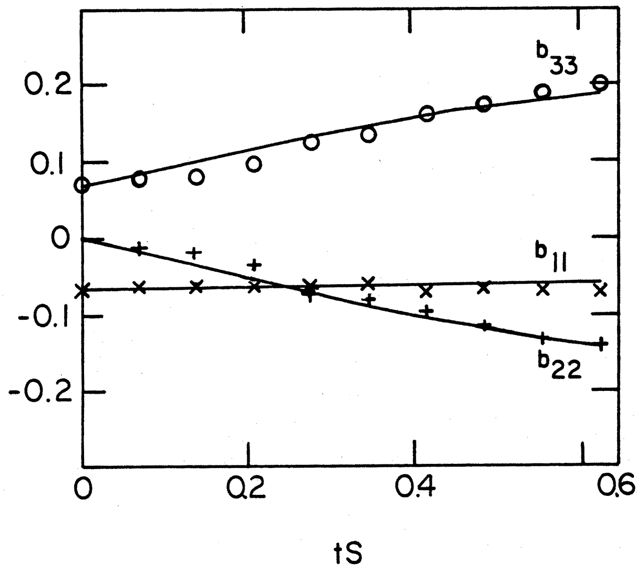

Fig. 2: Anisotropies b_{ij} against time t normalized by the mean strain rate S for the experiment of Gence and Mathieu[11]. Langevin model calculations (lines) Haworth and Pope[7], experimental data[11] (symbols).

If ε is known, then Eq. (8) is a closed equation which can be solved to determine the joint pdf. By comparing Eqs. 7 and 8 it may be seen that the Langevin model achieves closure by modelling the effects of the fluctuating pressure gradient and of viscous dissipation. But most importantly, convective transport and the mean pressure gradient are treated exactly: the left-hand sides of Eqs. 7 and 8 are identical. This is in marked contrast to moment closures in which turbulent convective transport has to be modelled.

For homogenous flows, Eq. 8 yields joint-normal solutions consistent with observations. For inhomogeneous flows, the equation can be solved numerically by a direct Monte Carlo method[12]. An example of such a Monte Carlo solution (though with different Lagrangian modelling) is given by Pope[13].

In summary, the generalized Langevin model is consistent with Kolmogorov's inertial range scaling laws, and accurately describes the evolution of the Reynolds stresses in homogeneous turbulence. It provides a closure to the Eulerian velocity pdf equation which can be solved, for inhomogeneous flows, by a Monte Carlo method.

Acknowledgements

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