# Consistent modeling of scalars in turbulent flows

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Conserved, passive scalars with equal diffusivities evolve independently according to the same linear transport equation. This observation leads to independence and linearity principles that impose constraints on the construction of consistent turbulence closure approximations. It is shown that the functional form of the scalar dissipation equation is more restrictive than has generally been assumed, and that the consistently modeled equation leads to qualitatively incorrect results for the decay of scalar fluctuations. It is also shown that the scalar-pressure-gradient correlation can be obtained from the velocity-pressure-gradient correlation, and thus an independent model for the scalar correlation is not required. (This last result depends upon an additional assumption that is certainly valid for nearly homogeneous flows.)

## I. INTRODUCTION

The development of turbulence models[1,2] has been aided by the principle of invariance,[3] which ensures the consistency of the model equations. A model that violates this principle erroneously produces different results in different coordinate systems. In this paper an analogous principle is developed for conserved passive scalars in turbulent flows.

In order to illustrate this principle, consider the flow of a coaxial jet into a surrounding fluid. The inner and outer jets carry conserved passive scalars (with equal diffusivities) denoted by $\phi_1$ and $\phi_2$. Turbulence model equations are available[1,4] to calculate statistics such as $\langle\phi_1\rangle$, $\langle\phi_2\rangle$, $\langle\phi_1'^2\rangle$, $\langle\phi_2'^2\rangle$, $\langle\phi_1'\phi_2'\rangle$, over the flow field. (Angled brackets and primes denote means and fluctuations, respectively.) The quantity

$$\phi_3 \equiv \phi_1 + \phi_2 \tag{1}$$

is also a conserved passive scalar. The variance $\langle\phi_3'^2\rangle$ can be calculated in two ways: directly, through the turbulence model equation for $\langle\phi_3'^2\rangle$; or indirectly through the model equations for $\langle\phi_1'^2\rangle$, $\langle\phi_2'^2\rangle$ and $\langle\phi_1'\phi_2'\rangle$ and the relation

$$\langle\phi_3'^2\rangle = \langle\phi_1'^2\rangle + \langle\phi_2'^2\rangle + 2\langle\phi_1'\phi_2'\rangle, \tag{2}$$

which follows simply from Eq. (1). Do the direct and indirect methods yield the same result? Only if the model equations are consistent with the principle developed here.

## II. LINEARITY AND INDEPENDENCE

We consider a set of $\sigma(\sigma \geqslant 2)$ conserved passive scalars denoted by $\boldsymbol{\phi} = \phi_1, \phi_2, \ldots, \phi_\sigma$. Each scalar satisfies the transport equation

$$\left(\rho\frac{\partial}{\partial t} + \rho U_i \frac{\partial}{\partial x_i} - \Gamma \frac{\partial^2}{\partial x_i\, \partial x_i}\right)\phi_\alpha = L(\phi_\alpha) = 0,$$
$$\alpha = 1,2,\ldots,\sigma, \tag{3}$$

where $\rho$ is the constant density, $\mathbf{U}(\mathbf{x},t)$ is the velocity of the turbulent flow field, and $\Gamma$ is the diffusivity—the same for each scalar. We assume the velocity to be known and different boundary and initial conditions to be applied to each scalar. Then the evolution of the scalars is uniquely determined by Eq. (3).

Since, by assumption, the scalars are passive, they affect either the density, the diffusivity, or the velocity. Thus the value of $\phi_\alpha(\mathbf{x},t)$ is unaffected by the values of $\phi_\beta$ $(\beta \neq \alpha)$ anywhere in the flow field. This is the independence principle.

Let $C_{\alpha\beta}$ be the components of a nonsingular but otherwise arbitrary $\sigma \times \sigma$ constant matrix. A different set of $\sigma$ scalars $\boldsymbol{\phi}^* = \phi_1^*, \phi_2^*, \ldots, \phi_\sigma^*$ is defined by the linear transformation

$$\phi_\alpha^* = C_{\alpha\beta}\phi_\beta, \tag{4}$$

where, here and below, summation is implied over repeated suffices. With $C_{\alpha\beta}^{-1}$ being the components of the inverse of C, the original scalars can be recovered by

$$\phi_\alpha = C_{\alpha\beta}^{-1}\phi_\beta^*. \tag{5}$$

Transformation rules for statistical quantities are readily obtained from Eq. (4). With angled brackets indicating a mean quantity, the velocity and scalar fluctuations are

$$u_i = U_i - \langle U_i\rangle, \tag{6}$$

and

$$\phi_\alpha' = \phi_\alpha - \langle\phi_\alpha\rangle. \tag{7}$$

The scalar variance (and covariance) $\langle\phi_\alpha'\phi_\beta'\rangle$ transforms by

$$\langle\phi_\alpha^{*\prime}\phi_\beta^{*\prime}\rangle = C_{\alpha\gamma}C_{\beta\eta}\langle\phi_\gamma'\phi_\eta'\rangle, \tag{8}$$

the scalar flux $\langle\phi_\alpha' u_i\rangle$ by

$$\langle\phi_\alpha^{*\prime}u_i\rangle = C_{\alpha\gamma}\langle\phi_\gamma' u_i\rangle, \tag{9}$$

and the scalar dissipation $\epsilon$

$$\epsilon_{\alpha\beta} \equiv \frac{2\Gamma}{\rho}\left\langle\frac{\partial\phi_\alpha'}{\partial x_i}\frac{\partial\phi_\beta'}{\partial x_i}\right\rangle, \tag{10}$$

transforms by

$$\epsilon_{\alpha\beta}^* = C_{\alpha\gamma}C_{\beta\eta}\epsilon_{\gamma\eta}. \tag{11}$$

The transformed scalars $\boldsymbol{\phi}^*$ satisfy the same transport equation as the scalars $\boldsymbol{\phi}$, since multiplying Eq. (3) by $C_{\beta\alpha}$ we obtain

$$C_{\beta\alpha}L(\phi_\alpha) = L(C_{\beta\alpha}\phi_\alpha) = L(\phi_\beta^*) = 0. \tag{12}$$

[This result depends, of course, on the fact that Eq. (3) is linear in $\phi_\alpha$.] As a consequence of this linearity, the transport equations for statistical quantities also obey the same transformation rules. For example, it follows from Eq. (9) and from the linearity of the operator

$$\overline{L} \equiv \left(\rho \frac{\partial}{\partial t} + \rho \langle U_i \rangle \frac{\partial}{\partial x_i} - \Gamma \frac{\partial^2}{\partial x_i \, \partial x_i}\right), \tag{13}$$

that the scalar flux equation transforms by

$$\overline{L}(\langle \phi_\gamma^{*\prime} u_i \rangle) = \overline{L}(C_{\gamma\alpha} \langle \phi'_\alpha u_i \rangle) = C_{\gamma\alpha} \overline{L}(\langle \phi'_\alpha u_i \rangle). \tag{14}$$

The modeled scalar-flux equation can be written

$$\overline{L}(\langle \phi'_\alpha u_i \rangle) = \mathscr{F}_{\alpha i}([\boldsymbol{\phi}]), \tag{15}$$

where $\mathscr{F}_{\alpha i}([\boldsymbol{\phi}])$ denotes a function of correlations of $\boldsymbol{\phi}$ (such as $\langle \phi'_\alpha u_i \rangle$, $\partial \langle \phi_\alpha \rangle / \partial x_i$, etc.). Since the modeling applies to any set of conserved passive scalars, for the transformed scalar flux equation we have

$$\overline{L}(\langle \phi_\gamma^{*\prime} u_i \rangle) = \mathscr{F}_{\gamma i}([\boldsymbol{\phi}^*]), \tag{16}$$

where $\mathscr{F}$ is the same function in Eqs. (15) and (16). From Eqs. (14)–(16) we obtain

$$\begin{aligned} \overline{L}(\langle \phi_\gamma^{*\prime} u_i \rangle) &= \mathscr{F}_{\gamma i}([\boldsymbol{\phi}^*]) \\ &= C_{\gamma\alpha} \overline{L}(\langle \phi'_\alpha u_i \rangle) = C_{\gamma\alpha} \mathscr{F}_{\alpha i}([\boldsymbol{\phi}]), \end{aligned} \tag{17}$$

that is,

$$\mathscr{F}_{\gamma i}([\boldsymbol{\phi}^*]) = C_{\gamma\alpha} \mathscr{F}_{\alpha i}([\boldsymbol{\phi}]). \tag{18}$$

Since C is an arbitrary matrix, Eq. (18) imposes stringent conditions on the form of the function $\mathscr{F}$ and on the admissible correlations $[\boldsymbol{\phi}]$. Specifically, $\mathscr{F}$ must be linear function of correlations (denoted by $q_{\alpha i}$) that transform as

$$q_{\gamma i}^* = C_{\gamma\alpha} q_{\alpha i}. \tag{19}$$

The quantities $\langle \phi'_\alpha u_i \rangle$ and $\partial \langle \phi_\alpha \rangle / \partial x_i$ satisfy this condition; so also do less likely quantities such as

$$g_{\alpha i} = \langle \phi'_\alpha \phi'_\beta \rangle^{-1} \epsilon_{\beta\gamma} \langle \phi'_\gamma u_i \rangle, \tag{20}$$

where $\langle \phi'_\alpha \phi'_\beta \rangle^{-1}$ are the components of the inverse of the covariance matrix. However, quantities such as $g_{\alpha i}$ which involve summation over all the scalars [the suffices $\beta$ and $\gamma$ in Eq. (20)] violate the independence principle. That is, the inclusion of a quantity such as $g_{\alpha i}$ in the scalar flux equation incorrectly implies that $\langle \phi'_\alpha u_i \rangle$ is affected by the other scalars $\phi_\beta, \beta \neq \alpha$.

A generalization of these considerations leads to the simple conclusion that scalar equations should be homogeneous (i.e., the same suffices should appear in each term, including any repeated suffices). This result can be used to guide the modeling of the equation for any scalar quantity—the scalar flux, the scalar variance, the scalar dissipation, or the scalar probability density function, for example. While this principle may appear to be obvious, its applications leads to two new and useful results.

## III. SCALAR DISSIPATION

In order to demonstrate the implications of the independence and linearity principles for the scalar dissipation equation, it is sufficient to consider statistically homogeneous scalars (without mean gradients) in homogeneous isotropic turbulence. Then the scalar dissipation equation is[15]

$$\frac{d\epsilon_{\alpha\beta}}{dt} = S_{\alpha\beta}, \tag{21}$$

where the source term $S_{\alpha\beta}$ is our concern here. In a second-order closure, $S_{\alpha\beta}$ can be modeled as the function $\mathscr{s}_{\alpha\beta}$,

$$S_{\alpha\beta} = \mathscr{s}_{\alpha\beta}(\langle \phi'_\gamma \phi'_\eta \rangle, \epsilon_{\gamma\eta}, \langle q^2 \rangle, \epsilon, \nu), \tag{22}$$

where $\langle q^2 \rangle$ is twice the turbulence kinetic energy

$$\langle q^2 \rangle = \langle u_l u_l \rangle, \tag{23}$$

$\epsilon$ is the rate of dissipation $\frac{1}{2}\langle q^2 \rangle$, and, for simplicity, the kinematic viscosity $\nu$ is assumed to be equal to $\Gamma / \rho$. By dimensional analysis and by invoking the linearity and independence principles, we can readily show that the most general expression for $S_{\alpha\beta}$ is

$$S_{\alpha\beta} = -a_1 \frac{\epsilon}{\langle q^2 \rangle} \epsilon_{\alpha\beta} + 2a_2 \left(\frac{\epsilon}{\langle q^2 \rangle}\right)^2 \langle \phi'_\alpha \phi'_\beta \rangle, \tag{24}$$

where $a_1$ and $a_2$ are scalar functions of the Reynolds number

$$R_l \equiv (\langle q^2 \rangle)^2 / (9 \epsilon \nu). \tag{25}$$

In order to study the yet simpler case of a single scalar variance we adopt the notation of Newman, Launder, and Lumley[5]:

$$\langle c^2 \rangle = \langle \phi'_1 \phi'_1 \rangle, \quad \epsilon_c = \tfrac{1}{2} \epsilon_{11}. \tag{26}$$

Then, the most general form of the equation for $\epsilon_c$ is

$$\frac{d\epsilon_c}{dt} = -a_1 \frac{\epsilon \epsilon_c}{\langle q^2 \rangle} + a_2 \left(\frac{\epsilon}{\langle q^2 \rangle}\right)^2 \langle c^2 \rangle. \tag{27}$$

It may be noted that the term

$$a_3 \epsilon_c^2 / \langle c^2 \rangle,$$

which has previously been suggested,[5] is inadmissible.

The form of Eq. (27) and the inadmissibility of the term in $a_3$ uncovers a serious problem with the scalar dissipation equation. The behavior of the scalar dissipation is best studied in terms of the nondimensional time scale ratio

$$r \equiv \frac{\epsilon_c / \langle c^2 \rangle}{\epsilon / \langle q^2 \rangle}. \tag{28}$$

Spalding[4] suggested that $r$ be taken as a universal constant ($r = 2.0$) and hence the need for an equation for $\epsilon_c$ is obviated. But the data of Lin and Lin[6] and those of Warhaft and Lumley[7] show that even in grid turbulence the time scale ratio is not a universal constant. Depending upon the initial conditions, values of $r$ are observed in the range $0.6 \leqslant r \leqslant 2.4$. Furthermore, it is found that the value of $r$ remains constant as the turbulence decays, rather than relaxing to an equilibrium value. A major aim of the work of Newman *et al.*[5] was to construct a modeled equation which (in accord with these observations) predicts that $r$ remains constant during decay.

For the consistently modeled equation for $\epsilon_c$ [Eq. (27)], and with the dissipation $\epsilon$ evolving by

$$\frac{d\epsilon}{dt} = \frac{-a_4 \epsilon^2}{\langle q^2 \rangle}, \tag{29}$$

the evolution equation for $r$ is

$$\frac{dr}{d\tau} = \dot{r}(r) = a_2 - a_5 r + 2r^2, \tag{30}$$

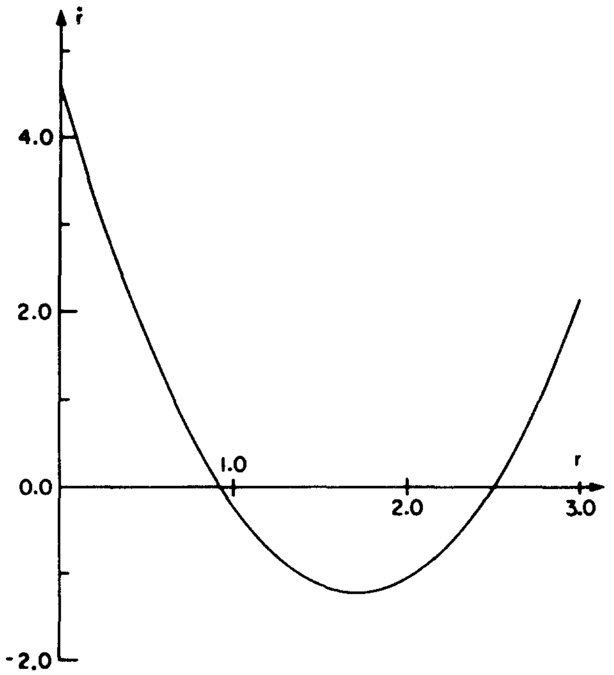


FIG. 1. Rate of change of $r$: $\dot{r}$ vs $r$, Eq. (30).

where $a_5 = 2 + a_1 - a_4$ and $\tau$ is the normalized time $d\tau = \epsilon\, dt/\langle q^2\rangle$. The experimental observation that $r$ remains constant suggests that $\dot{r}(r)$ should be zero—or very small—but, because of the quadratic term, Eq. (30) does not allow this. Figure 1 shows a plot of $\dot{r}(r)$ against $r$ for the values $a_2 = 4.630$ and $a_5 = 6.852$. Being quadratic, $\dot{r}(r)$ has two roots $r_e$ and $r_u$. If the initial value of $r$ is in the range $0 \leqslant r < r_u$ then as time proceeds $r$ tends towards the equilibrium value $r_e$. But if the initial value of $r$ exceeds $r_u$, then $r$ becomes infinite in finite time. Initial values of $r$ are observed in the range $0.6 \leqslant r \leqslant 2.4$. The constants $a_2$ and $a_5$ were selected by choosing $r_u = 2.5$ (so that, for all observed flows, $r$ does not become infinite) and by minimizing the maximum absolute values of $\dot{r}(r)$ in the range $0.6 \leqslant r \leqslant 2.4$. This procedure yields the values of $a_2$ and $a_5$ given above and

$$r_e = 0.9260, \tag{31}$$

$$|\dot{r}(r)| \leqslant \dot{r}(0.6) = -\dot{r}(1.713) = 1.239. \tag{32}$$

The behavior of the scalar dissipation equation Eq. (27) as reflected in the evolution of $r$ Eq. (30) is unsatisfactory. First, the existence in the model of a critical value $r_u$ (above which $r$ becomes infinite in finite time) makes it doubtful that the model equations have physically realizable solutions for all realizable initial conditions. Second, in the range of existing experimental data $0.6 \leqslant r \leqslant 2.4$, the value of $|\dot{r}|$ is of order one, whereas the data suggest that it is smaller by at least an order of magnitude.

It has been shown, then, that a consistent model of the scalar dissipation equation cannot reproduce the experimental observation that $r$ remains constant during decay. This does not mean that the observations defy theoretical explanation. Rather, the conclusion to be drawn is that the representation of the turbulence by only four quantities ($\langle q^2\rangle$, $\epsilon$, $\langle c^2\rangle$ and $\epsilon_c$) is inadequate. A fuller representation—perhaps in terms of spectra—is needed to determine the correct evolution of $\langle c^2\rangle$, $\epsilon_c$ and hence $r$.

## IV. SCALAR-PRESSURE-GRADIENT CORRELATION

The transport equation for the scalar flux $\langle \phi'_\alpha u_i\rangle$ contains, as an unknown to be modeled, the scalar-pressure-gradient correlation $\langle \phi'_\alpha\, \partial p/\partial x_i\rangle$, where $p$ is the fluctuating pressure. This correlation is generally modeled as a function of $\langle \phi'_\alpha u_i\rangle$, $\langle u_i u_j\rangle$, $\partial\langle U_i\rangle/\partial x_j$, and $\epsilon$ (see Refs. 8–10); and, in accord with the linearity and independence principle, it is generally modeled correctly as a linear function of $\langle \phi'_\alpha u_j\rangle$.

More information about the term is obtained by considering the modeling of the equivalent term in the velocity-scalar joint probability density function (pdf) equation.[11] Let $f(\mathbf{v},\boldsymbol{\psi})$ be the joint pdf of $\mathbf{u}$ and $\boldsymbol{\phi}'$ (where $\mathbf{v} = v_1, v_2, v_3$ and $\boldsymbol{\psi} = \psi_1, \psi_2, \ldots, \psi_\sigma$). Then the mean of any function of $\mathbf{u}$ and $\boldsymbol{\phi}'$, $Q(\mathbf{u},\boldsymbol{\phi}')$ is

$$\langle Q(\mathbf{u},\boldsymbol{\phi}')\rangle = \int\!\!\int Q(\mathbf{v},\boldsymbol{\psi}) f(\mathbf{v},\boldsymbol{\psi})\, d\mathbf{v}\, d\boldsymbol{\psi}. \tag{33}$$

Integration is from $-\infty$ to $\infty$ for each of the $3 + \sigma$ variables. The scalar-pressure-gradient correlation is then,

$$\left\langle \phi_\alpha \frac{\partial p}{\partial x_j}\right\rangle = \int\!\!\int \psi_\alpha \left\langle \frac{\partial p}{\partial x_j}\,\middle|\, \mathbf{v},\boldsymbol{\psi}\right\rangle f(\mathbf{v},\boldsymbol{\psi})\, d\mathbf{v}\, d\boldsymbol{\psi}, \tag{34}$$

where $\langle \partial p/\partial x_j | \mathbf{v},\boldsymbol{\psi}\rangle$ is the conditional expection of the pressure gradient given $\mathbf{u} = \mathbf{v}$ and $\boldsymbol{\phi}' = \boldsymbol{\psi}$.

In order to apply the independence principle to the modeling of the conditional pressure gradient, we define the following sets of scalars

$$\{\boldsymbol{\phi}\}_\sigma \equiv \phi_1, \phi_2, \ldots, \phi_\sigma, \tag{35}$$

$$\{\boldsymbol{\phi}\}_\omega \equiv \phi_1, \phi_2, \ldots, \phi_\omega, \quad (\omega > \sigma), \tag{36}$$

$$\{\boldsymbol{\phi}\}_{\sigma\omega} \equiv \phi_{\sigma+1}, \phi_{\sigma+2}, \ldots, \phi_\omega. \tag{37}$$

The general model for the pressure gradient conditional on $\mathbf{u} = \mathbf{v}$ and $\{\boldsymbol{\phi}\}_\sigma = \{\boldsymbol{\psi}\}_\sigma$ is

$$\left\langle \frac{\partial p}{\partial x_j}\,\middle|\, \mathbf{v},\{\boldsymbol{\psi}\}_\sigma\right\rangle = G_j(\{\boldsymbol{\psi}\}_\sigma, [\boldsymbol{\phi}]_\sigma, \mathbf{T}), \tag{38}$$

where $[\boldsymbol{\phi}]_\sigma$ denotes any statistic of $\{\boldsymbol{\phi}\}_\sigma$ (e.g., $\langle \phi'_\alpha \phi'_\beta\rangle$), and $\mathbf{T}$ is any tensor function of $\langle u_i u_j\rangle$, $\partial\langle U_i\rangle/\partial x_j$, $\rho$, and $\epsilon$. Since each scalar is independent and obeys the same transport equation, it would be inconsistent to apply different modeling to $\{\boldsymbol{\phi}\}_\omega$ than to $\{\boldsymbol{\phi}\}_\sigma$. Hence, for consistency with Eq. (38), we have

$$\left\langle \frac{\partial p}{\partial x_j}\,\middle|\, \mathbf{v},\{\boldsymbol{\psi}\}_\omega\right\rangle = G_j(\{\boldsymbol{\psi}\}_\omega, [\boldsymbol{\phi}]_\omega, \mathbf{T}). \tag{39}$$

Now, from the definition of conditional expectations, the two conditional pressure gradients are related by

$$\left\langle \frac{\partial p}{\partial x_j}\,\middle|\, \mathbf{v},\{\boldsymbol{\psi}\}_\sigma\right\rangle = \int \left\langle \frac{\partial p}{\partial x_j}\,\middle|\, \mathbf{v},\{\boldsymbol{\psi}\}_\omega\right\rangle f_{\sigma\omega}(\{\boldsymbol{\psi}\}_{\sigma\omega})\, d\{\boldsymbol{\psi}\}_{\sigma\omega}, \tag{40}$$

where $f_{\sigma\omega}$ is the joint pdf of $\{\boldsymbol{\phi}\}_{\sigma\omega}$, and integration is over the whole ($\omega$-$\sigma$)-dimensional space. Thus, combining these three equations we obtain a consistency requirement for the function $G_j$:

$$G_j(\{\psi\}_\sigma,[\phi]_\sigma,\mathbf{T}) = \int G_j(\{\psi\}_\omega,[\phi]_\omega,\mathbf{T}) f_{\sigma\omega}(\{\psi\}_{\sigma\omega}) d\{\psi\}_{\sigma\omega}. \quad (41)$$

It is obvious that this requirement is satisfied if $G_j$ is independent of the scalars since then

$$\left\langle \frac{\partial p}{\partial x_j} \middle| \mathbf{v},\{\psi\}_\omega \right\rangle = \left\langle \frac{\partial p}{\partial x_j} \middle| \mathbf{v},\{\psi\}_\sigma \right\rangle = \left\langle \frac{\partial p}{\partial x_j} \middle| \mathbf{v} \right\rangle. \quad (42)$$

But we hypothesize that there is no model (consistent with the linearity principle) that depends nontrivially on the scalars and satisfies this requirement, Eq. (41). While no general proof is offered, it is simply demonstrated that likely terms such as

$$\langle \phi'_\alpha \phi'_\beta \rangle^{-1} \langle u_j \phi'_\alpha \rangle \psi_\beta$$

do not satisfy Eq. (41). All possible terms involving the scalars that transform correctly contain the inverse of the covariance matrix. It is for this reason that, apparently, Eq. (41) cannot be satisfied. Thus we maintain that the conditional pressure gradient is independent of the scalars: reverting to the original notation,

$$\left\langle \frac{\partial p}{\partial x_j} \middle| \mathbf{v},\psi \right\rangle = \left\langle \frac{\partial p}{\partial x_j} \middle| \mathbf{v} \right\rangle. \quad (43)$$

The substitution of this result, Eq. (43), into Eq. (34) yields

$$\left\langle \phi'_\alpha \frac{\partial p}{\partial x_j} \right\rangle = \int \left\langle \frac{\partial p}{\partial x_j} \middle| \mathbf{v} \right\rangle \left\{ \int \psi_\alpha f(\mathbf{v},\psi) d\psi \right\} d\mathbf{v} = \int \left\langle \frac{\partial p}{\partial x_j} \middle| \mathbf{v} \right\rangle \left\langle \phi'_\alpha | \mathbf{v} \right\rangle f_u(\mathbf{v}) d\mathbf{v}, \quad (44)$$

where $f_u(\mathbf{v})$ is the joint pdf of the fluctuating velocities. The conditiontal expectation of $\phi'_\alpha$ given $\mathbf{u} = \mathbf{v}$ depends upon the shape of the joint pdf. Here we assume it to be given by

$$\langle \phi'_\alpha | \mathbf{v} \rangle = \langle u_k u_l \rangle^{-1} \langle u_l \phi'_\alpha \rangle v_k. \quad (45)$$

A sufficient (but not necessary) condition for this equation to be valid is that $\boldsymbol{\phi}'$ and $\mathbf{u}$ are joint-normally distributed. Measurements in simple shear flows[12,13] show this to be the case, to a good approximation. Substituting Eq. (45) into Eq. (44) we obtain the result

$$\left\langle \phi'_\alpha \frac{\partial p}{\partial x_j} \right\rangle = \langle u_k u_l \rangle^{-1} \langle u_l \phi'_\alpha \rangle \int \left\langle \frac{\partial p}{\partial x_j} \middle| \mathbf{v} \right\rangle v_k f(\mathbf{v}) d\mathbf{v} = \langle u_l \phi'_\alpha \rangle \langle u_k u_l \rangle^{-1} \left\langle u_k \frac{\partial p}{\partial x_j} \right\rangle. \quad (46)$$

Thus, with the assumption of Eq. (45) (which is certainly valid in the context of second-order closures) it has been shown that the scalar-pressure-gradient correlation can be determined from the velocity-pressure-gradient correlation by Eq. (46).

There are two sources of pressure fluctuations; one (denoted by $p^{(1)}$) due to the interaction of turbulence with the mean velocity gradients and the other $p^{(2)}$ solely due to the turbulence.[3,11] In Reynolds-stress closures, the term

$$\left\langle u_k \frac{\partial p^{(2)}}{\partial x_j} + u_j \frac{\partial p^{(2)}}{\partial x_k} \right\rangle,$$

is modeled. Because this is a combination of pressure gradients, Eq. (48) cannot be used to determine the corresponding model for $\langle \phi'_\alpha \partial p^{(2)}/\partial x_j \rangle$.

Pope[11] suggested the following model for the conditional gradient of $p^{(1)}$:

$$\left\langle \frac{\partial p^{(1)}}{\partial x_j} \middle| \mathbf{v},\psi \right\rangle = -2\rho \frac{\partial \langle U_l \rangle}{\partial x_m} C_{qmlj} v_q, \quad (47)$$

where C is a nondimensional function of the Reynolds stresses. By multiplying by $v_k$ and integrating we obtain

$$\left\langle u_k \frac{\partial p^{(1)}}{\partial x_j} \right\rangle = -2\rho \frac{\partial \langle U_l \rangle}{\partial x_m} C_{qmlj} \langle u_q u_k \rangle. \quad (48)$$

Now in Reynolds-stress closures, this term is generally modeled by

$$\left\langle u_k \frac{\partial p^{(1)}}{\partial x_j} \right\rangle = -\frac{1}{2} \rho \langle q^2 \rangle \frac{\partial \langle U_l \rangle}{\partial x_m} A_{jklm}, \quad (49)$$

where there are several suggestions for the tensor A (see Refs. 1 and 2). Comparing these two equations, we see that A and C are related by

$$A_{jklm} = (4\langle q^2 \rangle)^{-1} C_{qmlj} \langle u_k u_q \rangle, \quad (50)$$

and

$$C_{qmlj} = \tfrac{1}{4} \langle q^2 \rangle \langle u_k u_q \rangle A_{jklm}. \quad (51)$$

Thus the model Eq. (47) is consistent with existing Reynolds-stress models.

The corresponding model for $\langle \phi'_\alpha \partial p^{(1)}/\partial x_j \rangle$ can be obtained by multiplying Eq. (47) by $\psi_\alpha$ and integrating:

$$\left\langle \phi'_\alpha \frac{\partial p^{(1)}}{\partial x_j} \right\rangle = -2\rho \frac{\partial \langle U_l \rangle}{\partial x_m} C_{qmlj} \langle \phi'_\alpha u_q \rangle. \quad (52)$$

Alternatively the same result can be obtained from Eqs. (46) and (48). This shows that for a given pressure-gradient-velocity model (given C or A), the corresponding pressure-gradient-scalar model (Eq. (52) can be obtained. Previously,[3,9] the modeling of $\langle p^{(1)} \partial \phi'_\alpha / \partial x_j \rangle$ was viewed as a separate excercise, and the generally accepted model was the same as Eq. (52), but with C replaced by C*:

$$C^*_{qmlj} = \tfrac{2}{5} \delta_{qm} \delta_{lj} - \tfrac{1}{10} (\delta_{ql} \delta_{mj} + \delta_{qj} \delta_{lm}). \quad (53)$$

It can be shown that C* is the value of C in isotropic turbulence: Eqs. (50) and (52) provide a consistent model for the anisotropic case.


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