The motion of the core has to be considered. Asymptotic decay of the core flow during spindown to rest can be represented by $\Omega(t) = \Omega_i[1 + 0.69(2R/H)(\nu\Omega_i/R^2)^{1/2}t]^{-2}$, where Ω is the angular velocity.[10] It follows that the core has slowed down by about 9% by the time of Fig. 2(d). No coupling is observed in the cross-sectional view between the circular waves and the motion of the core. Therefore, it is concluded that the wave phenomenon is a property of the disk boundary layer. The number of observable cycles is limited by the size of the apparatus and the Reynolds number.

Observations made in a Bödewadt-type boundary layer occurring during impulsive spindown to rest in a cylindrical cavity show that a new class of circular waves are excited. These waves occur deep in the boundary layer and move toward the center. At sufficiently high Ekman numbers the well-known waves of type I are also excited along with the circular waves.

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A Lagrangian two-time probability density function equation for inhomogeneous turbulent flows

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An exact equation for the Lagrangian two-time velocity joint probability density function (pdf) is derived from the Navier–Stokes equation. The pdf equation contains as an unknown the conditional expectation of the fluid acceleration. A linear Markov model is proposed which leads to a modeled equation that is consistent both with Kolmogorov's theory in the inertial subrange and with Reynolds-stress models. The dissipation rate is obtained from the joint pdf in a way that is consistent with the modeled dissipation equation. A Monte Carlo method can be used to solve the modeled two-time pdf equation for inhomogeneous turbulent flows.

Several turbulent flow calculations have been reported based on one-point pdf methods.[1–3] In these methods a modeled transport equation is solved for the joint probability density function (pdf) of the three components of velocity $\mathbf{U}(\mathbf{x},t)$ at position $\mathbf{x}$ and time t. This approach has the advantage (compared to Reynolds-stress closures[4,5]) that convective transport appears in closed form and hence gradient-diffusion modeling is avoided.

The one-point joint pdf $f(\mathbf{V};\mathbf{x},t)$ (where V_1, V_2, and V_3 are the independent velocity variables) provides a complete statistical description of the velocity at each point and time, but it contains no joint information at two or more points. Consequently a time or length scale of turbulence cannot be deduced from f and (as with Reynolds-stress closures) scale information must be provided separately. This can be done either explicitly[1–3] or indirectly through the solution of a modeled transport equation for the rate of dissipation ϵ.[4,5] The direct specification of scale information is only possible in simple flows, and the validity and accuracy of the modeled dissipation equation has often been called in question (for example Ref. 6).

Multipoint pdf equations have been derived and modeled[7,8] but solutions have been obtained only for homogeneous isotropic turbulence.[7] The Monte Carlo methods,[9,10] by which the one-point pdf equations can be solved for inhomogeneous flows, cannot be simply extended to multipoint equations.

Here we consider a Lagrangian two-time pdf equation. This pdf contains both time-scale and length-scale information and its transport equation is amenable to solution by a Monte Carlo method. A simple model is proposed that is consistent both with Kolmogorov's theory in the inertial subrange and with Reynolds-stress models.

We consider a constant property turbulent flow (with unity density and viscosity μ) in which the Eulerian velocity $\mathbf{U}(\mathbf{x},t)$ satisfies the Navier–Stokes equation

$$\frac{\partial U_j}{\partial t} + U_i\frac{\partial U_j}{\partial x_i} = \mu\nabla^2 U_j - \frac{\partial p}{\partial x_j}, \qquad (1)$$

where $p(\mathbf{x},t)$ is the pressure. At time t the fluid particle at $\mathbf{x}$ has the Lagrangian position $\hat{\mathbf{x}}(t) = \mathbf{x}$ and velocity $\hat{\mathbf{U}}(t) = \mathbf{U}(\mathbf{x},t)$. At an earlier time s ($s < t$) the same fluid parti-

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cle has position $\hat{\mathbf{x}}(s)$ and velocity $\hat{\mathbf{U}}(s)$. We define the Lagrangian two-time pdf

$$g(\mathbf{V},\mathbf{W},\mathbf{y};\mathbf{x},t,s)$$

to be the probability density of the joint events

$$\hat{\mathbf{U}}(t) = \mathbf{V},\ \hat{\mathbf{U}}(s) = \mathbf{W},\ \hat{\mathbf{x}}(s) = \mathbf{y}. \tag{2}$$

Many important properties can be deduced from the definition of g: we note just two. The one-point joint pdf $f(\mathbf{V};\mathbf{x},t)$ is recovered by integrating g over all $\mathbf{W}$ and $\mathbf{y}$; and the Lagrangian two-time velocity correlation is obtained from

$$\langle \hat{U}_i(t)\hat{U}_j(s)\rangle = \iiint V_i W_j g\, d\mathbf{V}\, d\mathbf{W}\, d\mathbf{y}, \tag{3}$$

where integration is over all values $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{y}$. The Lagrangian integral time scale T can then be determined from $\langle \hat{U}_i(t)\hat{U}_j(s)\rangle$.

An exact evolution equation for g can be derived from the Navier–Stokes equation by standard methods.[10,11] The result is

$$\frac{\partial g}{\partial t} + V_i\frac{\partial g}{\partial x_i} = -\frac{\partial}{\partial V_i}\left(g\left\langle \mu\nabla^2 U_i - \frac{\partial p}{\partial x_i}\middle|\mathbf{Z}\right\rangle\right), \tag{4}$$

where, for any quantity, Q, $\langle Q|\mathbf{Z}\rangle$ denotes the expectation conditional upon the events (2). That is

$$\langle Q|\mathbf{Z}\rangle = \langle Q|\hat{\mathbf{U}}(t) = \mathbf{V},\ \hat{\mathbf{U}}(s) = \mathbf{W},\ \hat{\mathbf{x}}(s) = \mathbf{y}\rangle. \tag{5}$$

The terms on the left-hand side of Eq. (4) represent the rate of change and convection. They appear in closed form and hence no closure approximation is required. The right-hand side of the equation involves the conditional acceleration for which a model is required.

We now describe a simple model that is consistent both with Reynolds-stress closures and with Kolmogorov's inertial-range scaling laws. We do so via stochastic equations for the fluid particle properties that are similar to the Langevin equations.[12]

The first assumption is that g evolves by a Markov process. This allows us to replace the condition $\mathbf{Z}$ in Eq. (4) with the lesser condition $\hat{\mathbf{U}}(t) = \mathbf{V}$, since the conditions on $\hat{\mathbf{U}}(s)$ and $\hat{\mathbf{x}}(s)$ refer to the past. Next the right-hand side of Eq. (1) is decomposed into mean and fluctuating components

$$\mu\nabla^2 U_i - \frac{\partial p}{\partial x_i} = \mu\nabla^2\langle U_i\rangle - \frac{\partial\langle p\rangle}{\partial x_i} + a_i, \tag{6}$$

where

$$a_i = \mu\nabla^2 u_i - \frac{\partial p'}{\partial x_i}, \tag{7}$$

and u_i and p' are the fluctuating components of velocity and pressure. The evolution equation for g can now be written

$$\frac{\partial g}{\partial t} + V_i\frac{\partial g}{\partial x_i} + \left(\mu\nabla^2\langle U_i\rangle - \frac{\partial\langle p\rangle}{\partial x_i}\right)\frac{\partial g}{\partial V_i} = -\frac{\partial}{\partial V_i}(g\langle a_i|\mathbf{V}\rangle). \tag{8}$$

All the terms on the left-hand side are known in terms of g and hence require no modeling. It remains to model $\langle \mathbf{a}|\mathbf{V}\rangle$—the expectation of the fluctuating acceleration conditional upon $\mathbf{U}(\mathbf{x},t) = \mathbf{V}$.

In a small interval of time Δt the position and velocity of the fluid particle at $\hat{\mathbf{x}}(t)$ change by

$$\hat{\mathbf{x}}(t+\Delta t) = \hat{\mathbf{x}}(t) + \hat{\mathbf{U}}(t)\Delta t, \tag{9}$$

and

$$\hat{\mathbf{U}}(t+\Delta t) = \hat{\mathbf{U}}(t) + (\mu\nabla^2\langle\mathbf{U}\rangle - \nabla\langle p\rangle)\Delta t + \int_t^{t+\Delta t}\hat{\mathbf{a}}(t')dt', \tag{10}$$

where $\hat{\mathbf{a}}(t') = \mathbf{a}[\hat{\mathbf{x}}(t'),t']$. By analogy to Langevin's equation we model $\hat{\mathbf{a}}$ by a random contribution (isotropic white noise) and a deterministic contribution that is linear in the fluctuating velocity. Thus

$$\int_t^{t+\Delta t}\hat{a}_i(t')dt' = G_{ij}(\hat{U}_j - \langle U_j\rangle)\Delta t + (C_0\epsilon\Delta t)^{1/2}\xi_i, \tag{11}$$

where G_{ij} is a function of local mean quantities, C_0 is a universal constant, and $\boldsymbol{\xi}$ is a joint normal random vector with $\langle\boldsymbol{\xi}\rangle = 0$ and $\langle\xi_i\xi_j\rangle = \delta_{ij}$. (The specific form of G is discussed below.)

Equations (9)–(11) describe a random walk in position and velocity space. The corresponding pdf equation for $g(\mathbf{V},\mathbf{W},\mathbf{y};\mathbf{x},t,s)$ can be derived[12]:

$$\frac{\partial g}{\partial t} + V_i\frac{\partial g}{\partial x_i} + \left(\mu\nabla^2\langle U_i\rangle - \frac{\partial\langle p\rangle}{\partial x_i}\right)\frac{\partial g}{\partial V_i} = -\frac{\partial}{\partial V_i}\left(g\left[G_{il}(V_l - \langle U_l\rangle) - \frac{1}{2}C_0\epsilon\frac{\partial \ln g}{\partial V_i}\right]\right). \tag{12}$$

By comparing Eqs. (8) and (12) it may be seen that the term in square brackets [in Eq. (12)] is the corresponding model for the conditional acceleration $\langle a_i|\mathbf{V}\rangle$.

The model equation [Eq. (12)] is consistent with Kolmogorov's theory of local isotropy in the inertial subrange. For small time differences $t'\equiv t-s$, the structure function obtained from Eq. (12) is (to first order in t')

$$\langle[\hat{U}_i(s+t') - \hat{U}_i(s)]\,[\hat{U}_j(s+t') - \hat{U}_j(s)]\rangle = C_0\epsilon t'\delta_{ij}. \tag{13}$$

This result (first obtained by Obukhov[13]) is in accord with Kolmogorov's theory and identifies the universal constant C_0 (see Ref. 14, Eqs. 21.28–21.30′).

A modeled transport equation for the Reynolds stresses $\langle u_j u_k\rangle$ is obtained by multiplying Eq. (12) by $(V_j - \langle U_j\rangle)(V_k - \langle U_k\rangle)$ and integrating over all values of $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{y}$. The result is

$$\frac{\partial\langle u_j u_k\rangle}{\partial t} + \langle U_i\rangle\frac{\partial\langle u_j u_k\rangle}{\partial x_i} + \frac{\partial\langle u_i u_j u_k\rangle}{\partial x_i} + \langle u_i u_k\rangle\frac{\partial\langle U_j\rangle}{\partial x_i} + \langle u_i u_j\rangle\frac{\partial\langle U_k\rangle}{\partial x_i} = G_{kl}\langle u_l u_j\rangle + G_{jl}\langle u_l u_k\rangle + C_0\epsilon\delta_{jk}. \tag{14}$$

The tensor G can be chosen to make Eq. (14) [and hence Eq. (12)] consistent with any modeled Reynolds-stress equation. For example, for consistency with the model of Launder, Reece, and Rodi[4], G is given by

$$G_{ij} = 2C_{jmli}\frac{\partial \langle U_l \rangle}{\partial x_m} - \frac{1}{2}C_R\frac{\epsilon}{k}\delta_{ij} - C^*\epsilon\langle u_i u_j\rangle^{-1}, \quad (15)$$

where C_R is the Rotta constant, $C^* = 9C_0/2 - 3(C_R - 1)$, the tensor C is defined by Eq. (61) of Ref. 11, $\langle u_i u_j\rangle^{-1}$ are the components of the inverse of the Reynolds stress tensor, and the turbulent kinetic energy is $k = \langle u_i u_i \rangle/2$.

The modeled equation for g [Eq. (12)] with the specification of G [Eq. (15)] does not comprise a complete model since ϵ is as yet undetermined. There are many possible ways of extracting scale information from $g(\mathbf{V},\mathbf{W},\mathbf{y};\mathbf{x},t,s)$. A promising way is to relate the time scale

$$\tau = k/\epsilon, \quad (16)$$

to a damped Lagrangian time scale

$$\tau(t) = \frac{\alpha}{k}\int_{-\infty}^{t}\langle \hat{u}_j(t)\hat{u}_j(s)\rangle \exp\left(-\int_s^t \frac{\beta\, dt'}{\tau(t')}\right)ds, \quad (17)$$

where α and β are constants and

$$\hat{\mathbf{u}}(t) = \mathbf{u}[\hat{\mathbf{x}}(t),t]. \quad (18)$$

[It may be noted that with no damping ($\beta = 0$) and no decay ($dk/dt = 0$), Eq. (17) reduces to a constant relationship between τ and Lagrangian integral time scale: $\tau = 2\alpha T$.]

For the hypothetical case of homogeneous isotropic turbulence with isotropic energy production at the rate P, the model yields

$$\frac{d\epsilon}{dt} = \frac{\epsilon^2}{k}\left(\frac{C_{\epsilon 1}P}{\epsilon} - C_{\epsilon 2}\right), \quad (19)$$

where $C_{\epsilon 2}$ is an adjustable constant and $C_{\epsilon 1} \equiv 1.5$. This equation is identical to the empirical model equation for ϵ[4] for which the constant values are chosen to be $C_{\epsilon 1} = 1.44$ and $C_{\epsilon 2} = 1.92$.

The damping in Eq. (17) is added for computational reasons. It allows τ to be evaluated from

$$\tau(t) = (\alpha/k)\langle \hat{u}_j(t)\hat{r}_j(t)\rangle, \quad (20)$$

where $\hat{\mathbf{r}}(t)$ is the solution of the Lagrangian equation

$$\frac{d\hat{\mathbf{r}}}{dt} = \hat{\mathbf{u}} - \frac{\hat{\mathbf{r}}\beta}{\tau}. \quad (21)$$

An exact equation for the Lagrangian two-time velocity joint pdf has been derived from the Navier–Stokes equation. A multitime equation can readily be derived in the same way and the result is similar to Eq. (4).

A simple linear Markov model is proposed that is consistent both with Kolmogorov's theory in the inertial subrange and with Reynolds-stress closure schemes. The dissipation rate can be obtained from the joint pdf in a way that is consistent with the modeled dissipation equation.

The model equation can be solved by a Monte Carlo method[10] for inhomogeneous turbulent flows. The essence of the method is to solve Eqs. (9)–(11) and (21) to determine the properties $\hat{\mathbf{x}}(t)$, $\hat{\mathbf{U}}(t)$, and $\hat{\mathbf{r}}(t)$ of a large number of representative particles.

While a Markov model is presented here, the two-time pdf equation and the Monte Carlo method can accommodate non-Markovian models that may be more appropriate to inhomogeneous, nonstationary turbulent flows. A simple non-Markovian model results from making the tensor G a function of $\langle \hat{r}_i\hat{r}_j\rangle$ and $\langle \hat{r}_i\hat{u}_j\rangle$ (as well as of $\langle u_i u_j\rangle$, $\partial\langle U_i\rangle/\partial x_j$, and ϵ).

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