A stochastic Lagrangian model for acceleration in turbulent flows

Stephen B. Pope[a]
Sibley School of Mechanical and Aerospace Engineering, Cornell University, Ithaca, New York 14853

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A stochastic model is developed for the acceleration of a fluid particle in anisotropic and inhomogeneous turbulent flows. The model consists of an ordinary differential equation for velocity (which contains directly the acceleration due to the mean and rapid pressure gradients), and a stochastic model for the remainder of the acceleration, which is due to the slow pressure gradient and to viscosity. In addition to a rapid-pressure model, the stochastic model involves three tensor coefficients. For isotropic turbulence, the model reverts to that previously proposed by Sawford. At high Reynolds number the model is consistent with local isotropy and the Kolmogorov hypotheses, and tends to the generalized Langevin model for fluid-particle velocity. In this case two of the tensor coefficients are known in terms of the Kolmogorov constant $\mathcal{C}_0$, while the third is related to the coefficient in the generalized Langevin model. A complete analysis of the model is performed for homogeneous turbulent shear flow, for which there are Lagrangian data from direct numerical simulations. The main result is to establish the one-to-one correspondence between the model coefficients and the primary statistics, namely, the velocity and acceleration covariances and the tensor of velocity integral time scales. The autocovariances of velocity and acceleration obtained from the model are in excellent agreement with the direct numerical simulation (DNS) data. Future DNS studies of homogeneous turbulence can be used to investigate the dependence of the model coefficients on Reynolds number and on the imposed mean velocity gradients. The acceleration model can be used to generate a range of turbulence models which, in a natural way, incorporate Reynolds-number effects. [DOI: 10.1063/1.1483876]

I. INTRODUCTION

In order to investigate dispersion in turbulent flows, in 1921 Taylor[1] introduced a stochastic model for the position $\mathbf{X}^+(t)$ of a fluid particle. An analysis of Taylor's model shows that it is equivalent to the Langevin equation as a model for the fluid-particle velocity $\mathbf{U}^+(t)=d\mathbf{X}^+(t)/dt$. (Langevin[2] had proposed this stochastic equation in 1908 to model the velocity of particles undergoing Brownian motion.) The Langevin equation remains the basis for stochastic models of turbulent dispersion (see, e.g., Refs. 3–5). Furthermore the Langevin equation and its generalization[6,7] provide a closure to the transport equation for the (one-point, one-time) probability density function (PDF) of velocity.[8,9] And from the modeled velocity PDF equation can be deduced the corresponding partially modeled Reynolds-stress equation.[10] Thus, an accurate stochastic model for the fluid-particle velocity $\mathbf{U}^+(t)$ is a potent tool in turbulence modeling as well as in the study of turbulent dispersion.

Important conclusions about the performance of the Langevin model can be drawn from the simplest case of statistically stationary homogeneous isotropic turbulence. In general, the fluctuating component of fluid-particle velocity is defined by

$$\mathbf{u}^+(t)=\mathbf{U}^+(t)-\langle \mathbf{U}(\mathbf{X}^+[t],t)\rangle, \tag{1}$$

where $\mathbf{U}(\mathbf{x},t)$ is the Eulerian velocity; and for the case considered $\mathbf{u}^+(t)$ is a statistically stationary process with mean zero. The Lagrangian velocity autocorrelation function is defined by

$$\rho(s)\equiv\langle u^+_{(i)}(t)u^+_{(i)}(t+s)\rangle/\langle u^+_{(i)}(t)u^+_{(i)}(t)\rangle, \tag{2}$$

which is independent of t and i because of stationarity and isotropy, respectively. (Here and below, bracketed suffixes are excluded from the summation convention.) The Langevin model predicts this autocorrelation function to be[9]

$$\rho(s)=\exp\left(\frac{-|s|}{T_L}\right), \tag{3}$$

where T_L is the Lagrangian integral time scale. For not-too-small time intervals $|s|/T_L$, this prediction is in excellent agreement with experimental and direct numerical simulation (DNS) data.[11]

But the form of Eq. (3) reveals three related shortcomings of the Langevin model. First, it contains the single time scale T_L (which is characteristic of the large-scale, energy-containing motions); second, there is no dependence on Reynolds number; and, third, the slope of $\rho(s)$ given by Eq. (3) is discontinuous at the origin [reflecting the fact that the Langevin model for $\mathbf{u}^+(t)$ is continuous but not differentiable]. The same observations can be made[12] regarding the Lagrangian velocity frequency spectrum $E_L(\omega)$—which is the Fourier transform of $\langle u^+_{(i)}u^+_{(i)}\rangle\rho(s)$. According to the Langevin model, at high frequency $E_L(\omega)$ varies at ω^{-2}:

[a]Electronic mail: pope@mae.cornell.edu

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there is no representation of the more rapid decrease in $E_L(\omega)$ beyond the frequency corresponding to the Kolmogorov time scale τ_η.

In 1991, Sawford[12] introduced (for isotropic turbulence) a stochastic model for the fluid-particle acceleration $\mathbf{A}^+(t) = d\mathbf{U}^+(t)/dt = d^2\mathbf{X}^+(t)/dt^2$. Such a model remedies the above-mentioned deficiencies of the Langevin model: a second time scale (which scales with τ_η) is introduced; there is an intrinsic Reynolds-number dependence (since T_L/τ_η increases with Reynolds number); and, at the origin, the predicted velocity autocorrelation function is once continuously differentiable. Correspondingly, around the Kolmogorov frequency τ_η^{-1}, the Lagrangian velocity spectrum $E_L(\omega)$ smoothly changes its power-law behavior from ω^{-2} to ω^{-4}. For isotropic turbulence, Sawford's model is in excellent agreement with DNS data, including accounting for the Reynolds-number dependence of the acceleration autocorrelation function and the second-order Lagrangian velocity structure function.[12,13]

In this paper we consider a more general stochastic model for the fluid-particle acceleration, which is applicable to anisotropic turbulence and to inhomogeneous turbulent flows. The general form of the model is developed in Sec. II, where particular attention is paid to the contribution to acceleration from the rapid pressure gradient. When applied to homogeneous turbulence (with constant and uniform mean velocity gradients) the stochastic model is of the form

$$da_i^*(t) = -[C_{ij}a_j^*(t) + D_{ij}u_j^*(t)]dt + B_{ij}\,dW_j, \tag{4}$$

where $\mathbf{u}^*(t)$ is the model for $\mathbf{u}^+(t)$, $\mathbf{a}^*(t)$ is its rate of change (i.e., $\mathbf{a}^* \equiv d\mathbf{u}^*/dt$), and $\mathbf{W}(t)$ is an isotropic Wiener process.[9] The coefficients **B**, **C**, and **D** are tensors which can depend on the local state of the flow and the turbulence, but are independent of $\mathbf{a}^*$ and $\mathbf{u}^*$. (The conventional notation is that "+" denotes a fluid-particle property, and "*" denotes a model for that property.)

In the simplest case of isotropic turbulence, all the coefficients in Eq. (4) are isotropic (e.g., $B_{ij} = B\delta_{ij}$), and the model reverts to Sawford's.[12] In this case, which is reviewed in Sec. III C, there is a one-to-one correspondence between the three scalar coefficients (B, C, and D) and the three primary statistics: the acceleration variance a'^2; the velocity variance u'^2; and the velocity integral time scale T_L.

Beyond isotropic turbulence, the simplest type of flow to study is statistically stationary homogeneous turbulence with imposed mean velocity gradients—as exemplified by a recent DNS of forced homogeneous turbulent shear flow,[14] and described in Sec. IV. For this case the coefficients **B**, **C**, and **D** are constant, and a complete analysis of the model [Eq. (4)] can be performed. This is done in Sec. IV B, where it is shown that there is a one-to-one correspondence between the tensor coefficients in the model and the primary statistics, namely the velocity-acceleration covariances and the velocity integral time scale tensor. (This analysis parallels the authors's recent analysis of a stochastic model for velocity.[15])

With some approximation (and with an appropriate scaling of the variables), the same analysis can be applied to nonstationary homogeneous turbulence, in particular to (unforced) homogeneous turbulent shear flow for which there are Lagrangian data from the recent DNS studies of Sawford and Yeung.[16,17] It is shown (in Sec. IV C) that the velocity-acceleration autocorrelation functions predicted by the model are in excellent agreement with these DNS data.

As well as being useful in its own right, we also regard the acceleration model as an intermediate step in the development of improved stochastic models for velocity for use in dispersion studies, in PDF methods, and in other turbulence models. Compared to the velocity model, the acceleration model can be more closely related to Lagrangian data from DNS, which are known to contain strong Reynolds-number dependencies.[11,18] Given an acceleration model (i.e., a prescription for the coefficients **B**, **C**, and **D**), a corresponding velocity model can be deduced[15] which yields the same velocity covariances and integral time scales, and which inherits its Reynolds-number dependencies. Such an improved model has direct application in PDF methods, and from it can be deduced a pressure-rate-of-strain model for use in Reynolds-stress models. These and other uses of the acceleration model are discussed in Sec. V.

II. STOCHASTIC MODEL FOR ACCELERATION

We consider the inhomogeneous turbulent flow of a constant-property Newtonian fluid (of density ρ and kinematic viscosity ν). This is governed by the continuity equation $\partial U_i/\partial x_i = 0$, and the Navier–Stokes equation

$$A_i(\mathbf{x},t) \equiv \frac{DU_i}{Dt} = \left(\frac{\partial}{\partial t} + U_j\frac{\partial}{\partial x_j}\right)U_i = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 U_i}{\partial x_j \partial x_j}, \tag{5}$$

where $\mathbf{A}(\mathbf{x},t)$, $\mathbf{U}(\mathbf{x},t)$, and $p(\mathbf{x},t)$ are the acceleration, velocity, and pressure. The general fluid particle has position $\mathbf{X}^+(t)$, velocity,

$$\mathbf{U}^+(t) = \frac{d\mathbf{X}^+(t)}{dt} = \mathbf{U}(\mathbf{X}^+[t],t), \tag{6}$$

and acceleration

$$\mathbf{A}^+(t) = \frac{d\mathbf{U}^+(t)}{dt} = \mathbf{A}(\mathbf{X}^+[t],t). \tag{7}$$

A. Decomposition of acceleration

The acceleration can be decomposed into mean and fluctuating contributions based on the mean ($\langle\mathbf{U}\rangle$ and $\langle p\rangle$) and fluctuating ($\mathbf{u}$ and p') components of velocity and pressure. Furthermore, as originally shown by Chou,[19] the fluctuating pressure can be decomposed into rapid, $p^{(r)}$, slow, $p^{(s)}$, and harmonic, $p^{(h)}$, contributions.[9] Thus, the fluid acceleration is

$$A_i = -\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_i} - \frac{1}{\rho}\frac{\partial p^{(r)}}{\partial x_i} - \frac{1}{\rho}\frac{\partial p^{(s)}}{\partial x_i} - \frac{1}{\rho}\frac{\partial p^{(h)}}{\partial x_i} + \nu\frac{\partial^2\langle U_i\rangle}{\partial x_j\partial x_j} + \nu\frac{\partial^2 u_i}{\partial x_j \partial x_j}. \tag{8}$$

The harmonic pressure and the mean viscous term are negligible except in the immediate vicinity of walls (or other surfaces). Here we neglect these terms, and hence leave to future work the development of the special treatments required for the viscous near-wall region.

B. Structure of the model

The proposed model consists of an ordinary differential equation (ODE) for $\mathbf{U}^*(t)$—a model for the fluid particle velocity $\mathbf{U}^+(t)$—and a stochastic differential equation (SDE) for an acceleration variable denoted by $\mathbf{A}^0(t)$. The model also involves the fluctuating components of these quantities, which are defined by

$$\mathbf{u}^*(t) \equiv \mathbf{U}^*(t) - \langle \mathbf{U}^*(t) | \mathbf{X}^*(t) \rangle \tag{9}$$

and

$$\mathbf{a}^0(t) \equiv \mathbf{A}^0(t) - \langle \mathbf{A}^0(t) | \mathbf{X}^*(t) \rangle, \tag{10}$$

where $\mathbf{X}^*(t)$ denotes the position of the model particle.

The ODE for velocity is

$$\frac{dU_i^*}{dt} = -\left(\frac{1}{\rho}\frac{\partial \langle p \rangle}{\partial x_i}\right)_{\mathbf{X}^*(t)} - \left(\frac{1}{\rho}\frac{\partial p^{(r)}}{\partial x_i}\right)_{\mathbf{X}^*(t)} + a_i^0(t), \tag{11}$$

where the first two contributions on the right-hand side represent acceleration by the mean pressure gradient (which is assumed to be known), and acceleration by the rapid pressure gradient (which has to be modeled). A comparison of Eq. (8) and Eq. (11) then reveals that $\mathbf{a}^0(t)$ is a model for the acceleration due to the slow pressure gradient and the viscous term.

The acceleration variable $\mathbf{A}^0(t)$ is modeled by the general SDE,

$$dA_i^0(t) = -[C_{ij}A_j^0(t) + D_{ij}u_j^*(t)]dt + B_{ij}\, dW_j, \tag{12}$$

where $\mathbf{W}(t)$ is an isotropic Wiener process. The tensor functions $\mathbf{B}(\mathbf{x},t)$, $\mathbf{C}(\mathbf{x},t)$, and $\mathbf{D}(\mathbf{x},t)$ [which in Eq. (12) are evaluated at $[\mathbf{X}^*(t),t]$] depend on the local state of the turbulence, but are independent of $\mathbf{U}^*$ and $\mathbf{A}^0$.

C. Homogeneous turbulence

Before presenting the rationale for the structure of the model, we first note the form that it takes in homogeneous turbulence.

In homogeneous turbulence (with uniform mean velocity gradients), the coefficients $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ depend only on time, and it follows that the mean $\langle \mathbf{A}^0(t) | \mathbf{X}^*(t) \rangle = \langle \mathbf{A}^0(t) \rangle$ is zero. Consequently $\mathbf{a}^0(t)$ is identical to $\mathbf{A}^0(t)$. And the velocity equation can readily be transformed to an equation for $\mathbf{u}^*(t)$. Thus, for homogeneous turbulence the model becomes

$$\frac{du_i^*}{dt} = -\frac{\partial \langle U_i \rangle}{\partial x_j} u_j^* - \left(\frac{1}{\rho}\frac{\partial p^{(r)}}{\partial x_i}\right)_{\mathbf{X}^*(t)} + a_i^0(t), \tag{13}$$

$$da_i^0(t) = -[C_{ij}a_j^0(t) + D_{ij}u_j^*(t)]dt + B_{ij}\, dW_j. \tag{14}$$

D. Rationale

The structure of the model is such that some contributions to acceleration—namely, from the mean and rapid pressure gradients—appear directly in the ODE for velocity, Eq. (11); whereas the other contributions—from the slow pressure gradient and viscosity—are modeled through the SDE for $\mathbf{A}^0(t)$, Eq. (12). The rationale for this division is based on the response of the system to a rapid distortion, and it can be most easily understood for the case of homogeneous turbulence.

Consider the sudden imposition of a very large strain rate on homogeneous turbulence. Both the mean and rapid pressure fields change suddenly and this leads to a sudden change in the fluid acceleration. On the other hand, the fluctuating velocity field and the slow pressure change continuously in response to the suddenly imposed distortion.

The model is qualitatively in accord with this behavior. It may be seen from Eq. (11) and Eq. (13) that the acceleration changes suddenly if there is a sudden change in $\partial\langle U_i \rangle/\partial x_j$, with accompanying sudden changes in $\partial\langle p \rangle/\partial x_i$ and $\partial p^{(r)}/\partial x_i$. In the acceleration equation [Eq. (12) and Eq. (14)], these sudden changes can result in sudden changes in the coefficients, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$, but nevertheless, $\mathbf{a}^0(t)$ changes continuously.

E. Rapid-pressure models

As is usual, and in keeping with the physics, we consider deterministic models for the rapid pressure gradient. The quantity then to be modeled is the conditional mean rapid pressure gradient—conditional on the modeled state of the fluid particle.

For homogeneous turbulence, the rapid pressure varies linearly with the imposed mean velocity gradient, and hence the general model can be written[20]

$$\left\langle -\frac{1}{\rho}\frac{\partial p^{(r)}}{\partial x_i} \middle| \mathbf{a}^0, \mathbf{u}^* \right\rangle = 2\frac{\partial \langle U_k \rangle}{\partial x_\ell} N_{\ell k i}. \tag{15}$$

The third-order tensor function $N_{\ell k i}$ is given in terms of two-point conditional velocity statistics in Refs. 20 and 8 (where it is denoted by $B_{\ell k i}$), and it satisfies the relations

$$N_{\ell i i} = u_\ell^*, \quad N_{\ell \ell i} = 0, \quad N_{\ell k i} = N_{\ell i k}. \tag{16}$$

Rapid distortions of homogeneous turbulence can be treated exactly using the wave-vector model of Van Slooten and Pope.[9,21] This requires that the modeled state of the fluid particle be supplemented by the wave vector $\mathbf{e}^*(t)$—which, among other conditions, satisfied the relations

$$e_i^* e_i^* \equiv 1, \quad e_i^* u_i^* = 0. \tag{17}$$

Then the tensor $N_{\ell k i}$ in Eq. (15) is given by

$$N_{\ell k i} = u_\ell^* e_k^* e_i^*. \tag{18}$$

For rapid distortions, the wave-vector model consists of ODE's for $\mathbf{e}^*(t)$ and $\mathbf{u}^*(t)$, the latter being Eq. (13) with the neglect of $\mathbf{a}^0$, and with the rapid-pressure model given by Eqs. (15) and (18). This model is exact for arbitrary rapid distortions of homogeneous turbulence, in the sense that it yields the correct evolution of the Reynolds stresses.

As is conventional in Reynolds-stress and velocity-PDF modeling, we are primarily concerned here with models based on velocity and its one-point statistics, i.e., $\mathbf{u}^*(t)$ and the Reynolds stress $\langle u_i u_j \rangle$. The unfortunate fact of the matter is that these quantities are inadequate to describe rapid distortions (see, e.g., Reynolds and Kassinos[22]): additional directional information is needed, as is provided by the wave vector. However, the hope is that rapid-pressure models based on velocity alone may be adequate for the moderate and slowly varying mean strain rates encountered in many turbulent shear flows.

Following Ref. 20, it is natural to consider a rapid-pressure model that is linear in velocity, and which therefore can be written

$$\left\langle -\frac{1}{\rho}\frac{\partial p^{(r)}}{\partial x_i}\middle| \mathbf{a}^0, \mathbf{u}^* \right\rangle = G_{ij}^{(r)} u_j^* = H_{ijk\ell}^{(r)} u_j^* \frac{\partial \langle U_k \rangle}{\partial x_\ell}, \tag{19}$$

or, equivalently,

$$N_{\ell k i} = \tfrac{1}{2} H_{ijk\ell}^{(r)} u_j^*, \tag{20}$$

where the tensors $G_{ij}^{(r)}$ and $H_{ijk\ell}^{(r)}$ correspond to the analogous tensors in the Haworth–Pope model.[7,9] The nondimensional fourth-order tensor $\mathbf{H}^{(r)}$ is modeled as a linear function of the Reynolds-stress anisotropy tensor

$$b_{ij} \equiv \frac{\langle u_i u_j \rangle}{\langle u_k u_k \rangle} - \frac{1}{3}\delta_{ij}, \tag{21}$$

and indeed a nontrivial dependence on b_{ij} is required to satisfy the condition that the rapid pressure neither produces nor removes turbulent kinetic energy.

In subsequent sections we confine attention to this linear model, not least because it is amenable to analysis. In DNS, the rapid pressure gradient can extracted, its linearity in $\mathbf{u}^+$ can be examined, and specific models for $H_{ijk\ell}^{(r)}$ can be assessed. With this model, Eq. (13) can be rewritten

$$\frac{du_i^*}{dt} = -K_{ij} u_j^* + a_i^0, \tag{22}$$

where the tensor K_{ij} is defined by

$$K_{ij} = \frac{\partial \langle U_i \rangle}{\partial x_j} - G_{ij}^{(r)} = \frac{\partial \langle U_k \rangle}{\partial x_\ell}[\delta_{ik}\delta_{j\ell} - H_{ijk\ell}^{(r)}]. \tag{23}$$

Finally, we caution that in future studies of rapid-pressure modeling (e.g., based on DNS) nonlinear models should not be discounted. For example, the simple model

$$N_{\ell k i} = \frac{1}{2} u_\ell \left(\delta_{ik} - \frac{u_i^* u_k^*}{u_j^* u_j^*} \right), \tag{24}$$

satisfies all known constraints (without requiring a dependence on b_{ij}).

F. Summary

The model consists of an ODE for velocity, Eq. (11), which contains the mean pressure gradient and a model for the rapid pressure gradient [e.g., Eq. (19)]. The remainder of the acceleration—owing to the slow pressure gradient and the viscous term—is modeled by an SDE, Eq. (12), which contains three tensor coefficients, **B**, **C**, and **D**. Various properties of these coefficients are revealed in subsequent sections.

III. PROPERTIES OF THE MODEL

In this section we examine some of the mathematical properties of the model, and their connections to the physics of turbulent motions.

A. Equivalent first-order and second-order systems

For homogeneous turbulence, the model [Eq. (14) and Eq. (22)] can be written as a first-order system of SDE's,

$$du_i^* = [-K_{ij} u_j^* + a_i^0] dt, \tag{25}$$

$$da_i^0 = -[C_{ij} a_j^0 + D_{ij} u_j^*] dt + B_{ij}\, dW_j, \tag{26}$$

or, in an inferior notation, as a first-order system of ODE's

$$\frac{du_i^*}{dt} = -K_{ij} u_j^* + a_i^0, \tag{27}$$

$$\frac{da_i^0}{dt} = -C_{ij} a_j^0 - D_{ij} u_j^* + B_{ij} \dot{W}_j, \tag{28}$$

where $\dot{\mathbf{W}}$ denotes white noise, which has the property $\int_0^t \dot{\mathbf{W}}(t')dt' = \mathbf{W}(t)$.

Alternatively, by differentiating Eq. (27) with respect to t, the model can be re-expressed as the second-order system

$$\frac{d^2 u_i^*}{dt^2} + (C_{ij} + K_{ij})\frac{du_j^*}{dt} + \left(D_{ij} + \frac{dK_{ij}}{dt} \right) u_j^* = B_{ij} \dot{W}_j. \tag{29}$$

It may be seen that the system is governed fundamentally by just three coefficient tensors, not four as suggested by the appearance of **B**, **C**, **D**, and **K** in Eqs. (25)–(28). In particular, if **K** is constant—as is the case in the analysis below (Sec. IV B)—the behavior of $\mathbf{u}^*(t)$ is determined by **B**, **D** and the sum

$$\overline{\mathbf{C}} \equiv \mathbf{C} + \mathbf{K}, \tag{30}$$

but not by **C** and **K** individually. Thus, for constant **K**, Eqs. (25) and (26) are equivalent to the system

$$du_i^* = a_i^*\, dt, \tag{31}$$

$$da_i^* = -[\overline{C}_{ij} a_j^* + D_{ij} u_j^*] dt + B_{ij}\, dW_j, \tag{32}$$

in which Eq. (31) defines $\mathbf{a}^* \equiv d\mathbf{u}^*/dt$, and $\mathbf{a}^0$ can be recovered as

$$a_i^0 = a_i^* + K_{ij} u_j^*. \tag{33}$$

The model is analyzed below via Eqs. (31) and (32).

B. Scaled model equations and coefficients

It is informative to scale the variables and coefficients in the model equations for homogeneous turbulence so that they become nondimensional quantities of order unity. A preliminary is to define the quantities used to perform these scalings.

The velocity and acceleration variables are scaled by their standard deviations u' and a', which are given by

$$u'^2 \equiv \tfrac{1}{3}\langle u_i^* u_i^* \rangle = \tfrac{2}{3}k, \quad a'^2 \equiv \tfrac{1}{3}\langle a_i^* a_i^* \rangle, \tag{34}$$

TABLE I. Summary of different time scales.

$\tau \equiv k/\epsilon$	Turbulence time scale
$\tau_\eta \equiv (\nu/\epsilon)^{1/2}$	Kolmogorov time scale
$\tau_\mathcal{S} \equiv \mathcal{S}^{-1}$	Shear time scale
$T_L \equiv \frac{1}{3}\hat{T}^u_{ii}$	Lagrangian velocity integral time scale
$\tau_a \equiv u'^2/(a'^2\tau)$	Acceleration time scale
$T_\infty \equiv \lambda_1^{-1}$, Eq. (55)	Velocity eigen-time scale
$\tau_0 \equiv \lambda_2^{-1}$, Eq. (57)	Acceleration eigen-time scale
$\mathbf{T}$, Eq. (102)	Integral time scale (6×6) matrix
$\hat{\mathbf{T}}^{uu}$, Eq. (109)	Velocity integral time scale tensor

where k is the turbulent kinetic energy. There are four relevant time scales. The turbulence time scale is defined by

$$\tau \equiv \frac{k}{\epsilon}, \tag{35}$$

where ϵ is the rate of dissipation of k. The shear time scale, characteristic of the imposed mean velocity gradients, is defined by

$$\tau_\mathcal{S} \equiv \mathcal{S}^{-1}, \tag{36}$$

where

$$\mathcal{S}^2 \equiv \frac{\partial\langle U_i\rangle}{\partial x_j}\frac{\partial\langle U_i\rangle}{\partial x_j}. \tag{37}$$

The Kolmogorov time scale is

$$\tau_\eta \equiv \left(\frac{\nu}{\epsilon}\right)^{1/2}, \tag{38}$$

and the acceleration time scale is defined by

$$\tau_a \equiv \frac{u'^2}{a'^2\tau}. \tag{39}$$

The ratio τ_η/τ decreases with Reynolds number as

$$\frac{\tau_\eta}{\tau} = \mathrm{Re}^{-1/2} = \left(\frac{20}{3}\right)^{1/2} R_\lambda^{-1}, \tag{40}$$

where the turbulence Reynolds number is $\mathrm{Re} \equiv k^2/(\epsilon\nu)$, and the Taylor-scale Reynolds number is $R_\lambda \equiv (\frac{20}{3}\mathrm{Re})^{1/2}$. The various time scales used throughout the paper are summarized in Table I.

With a_0 being the Kolmogorov-scaled acceleration variance

$$a_0 \equiv \frac{a'^2\tau_\eta}{\epsilon}, \tag{41}$$

the acceleration and Kolmogorov time scales are related by

$$\frac{\tau_a}{\tau_\eta} = \frac{2}{3a_0}. \tag{42}$$

It may be seen then that (at least approximately at high Reynolds number) τ_a scales with τ_η, since according to the Kolmogorov hypotheses a_0 is a universal constant.[23] In fact, it is known[11,24–26] that, at moderate Reynolds numbers, a_0 increases weakly with R_λ—in accord with the refined Kolmogorov hypotheses. In discussing scalings we ignore this weak dependence and write $\tau_a/\tau \sim \mathrm{Re}^{-1/2}$.

The ODE for velocity, Eq. (13) and Eq. (15), can be written

$$\frac{1}{u'}\frac{du_i^*}{d\hat{t}} = \frac{\tau}{u'}\frac{du_i^*}{dt} = \frac{\tau}{\tau_\mathcal{S}}\left[\tau_\mathcal{S}\frac{\partial\langle U_k\rangle}{\partial x_\ell}\right]\left[\frac{2N_{\ell ki}}{u'} - \frac{u_\ell^*}{u'}\delta_{ik}\right] + \left(\frac{\tau}{\tau_a}\right)^{1/2}\left[\frac{a_i^0}{a'}\right]. \tag{43}$$

Each term is nondimensional; expressions in square brackets are of order unity; and time is scaled by the turbulence time scale, i.e., $\hat{t} \equiv t/\tau$. If the linear rapid-pressure model, Eq. (19), is used, then the ODE for $\mathbf{u}^*(t)$ can alternatively be written

$$\frac{1}{u'}\frac{du_i^*}{d\hat{t}} = -\frac{\tau}{\tau_\mathcal{S}}\tilde{K}_{ij}\left[\frac{u_j^*}{u'}\right] + \left(\frac{\tau}{\tau_a}\right)^{1/2}\left[\frac{a_i^0}{a'}\right], \tag{44}$$

where the nondimensional, order-one coefficient $\tilde{\mathbf{K}}$ is

$$\tilde{K}_{ij} = \tau_\mathcal{S} K_{ij} = \tau_\mathcal{S}\frac{\partial\langle U_k\rangle}{\partial x_\ell}[\delta_{ik}\delta_{j\ell} - H^{(r)}_{ijk\ell}]. \tag{45}$$

It is clear from Eqs. (43) and (44) that $\mathbf{u}_i^*/u'$ responds to the mean velocity gradients at the normalized rate $\tau/\tau_\mathcal{S} = \mathcal{S}k/\epsilon$. Under usual circumstances this is of order one, but for rapid distortions it is arbitrarily large. Evidently, the term in $\mathbf{a}^0$ is of order $\sqrt{\tau/\tau_a} \sim \mathrm{Re}^{1/4}$. But since $\mathbf{a}^0$ is a zero-mean random function with normalized time scale τ_a/τ, the cumulative effect of the term on the covariances of $\mathbf{u}^*/u'$ over a time interval $\Delta\hat{t} \gg \tau_a/\tau$ is of order $\sqrt{\tau/\tau_a}^2(\Delta\hat{t}\tau_a/\tau) \sim \Delta\hat{t}$. Thus, although the term in $\mathbf{a}^0$ is relatively large instantaneously (of order $\mathrm{Re}^{1/4}$), its cumulative effect is of order one.

For the SDE for $\mathbf{a}^0(t)$, Eq. (14), we define the scaled coefficients by

$$\tilde{\mathbf{B}}^2 = \frac{\tau_a}{a'^2}\mathbf{B}^2, \quad \tilde{\mathbf{C}} = \tau_a\mathbf{C}, \quad \tilde{\mathbf{D}} = \tau\tau_a\mathbf{D}. \tag{46}$$

The subsequent analysis confirms that these scalings are appropriate, in that each of these scaled coefficients is of order unity. With these definitions, Eq. (14) can be written

$$\frac{da_i^0}{a'} = -\left[\tilde{C}_{ij}\frac{a_j^0}{a'} + \left(\frac{\tau_a}{\tau}\right)^{1/2}\tilde{D}_{ij}\frac{u_j^*}{u'}\right]\frac{dt}{\tau_a} + \tilde{B}_{ij}\frac{dW_j}{\sqrt{\tau_a}}. \tag{47}$$

Clearly τ_a is the characteristic time scale of the process: the mean of the term in $\tilde{\mathbf{C}}$, and the variance of the term in $\tilde{\mathbf{B}}$ is each of order dt/τ_a. However, the term in $\tilde{\mathbf{D}}$ is smaller by the factor of $(\tau_a/\tau)^{1/2} \sim \mathrm{Re}^{-1/4}$.

If the mean velocity gradients are constant, then the equations for $\mathbf{u}^*(t)$ and $\mathbf{a}^0(t)$ [Eqs. (25) and (26)] can be re-expressed as equations for $\mathbf{u}^*(t)$ and $\mathbf{a}^*(t)$ [Eqs. (31) and (32)]. The scaled forms of these equations are

$$\frac{\tau}{u'}\frac{du_i^*}{dt} = \left(\frac{\tau}{\tau_a}\right)^{1/2}\frac{a_i^*}{a'} \tag{48}$$

and

$$\frac{da_i^*}{a'} = -\left[\tilde{C}_{ij}\frac{a_j^*}{a'} + \left(\frac{\tau_a}{\tau_S}\right)\tilde{K}_{ij}\frac{a_j^*}{a'} + \left(\frac{\tau_a}{\tau}\right)^{1/2}\tilde{D}_{ij}\frac{u_j^*}{u'}\right]\frac{dt}{\tau_a} + \tilde{B}_{ij}\frac{dW_j}{\sqrt{\tau_a}}. \quad (49)$$

For the case considered τ/τ_S is of order unity, so that (compared to the leading-order terms) the terms in $\tilde{\mathbf{K}}$ and $\tilde{\mathbf{D}}$ are of order $\tau_a/\tau \sim \mathrm{Re}^{-1/2}$ and $(\tau_a/\tau)^{1/2} \sim \mathrm{Re}^{-1/4}$, respectively.

C. Isotropic turbulence

We consider in this section the simplest case of homogeneous isotropic turbulence made statistically stationary by artificial forcing. We do so to relate the general model proposed here to Sawford's,[12] and to provide a characterization of the model's behavior in this simple setting. This provides a useful reference for the results obtained below for the general case.

For isotropic turbulence without mean velocity gradients, there is no rapid pressure, and $\mathbf{a}^*(t) = d\mathbf{u}^*/dt$ is the model for the fluid-particle acceleration. The coefficients in the model, Eq. (32), are inevitably isotropic ($B_{ij} = B\delta_{ij}$, $\tilde{B}_{ij} = \tilde{B}\delta_{ij}$, etc.), and so the three components of $\mathbf{a}^*(t)$ are statistically identical and independent. Writing $a^*(t)$ for one component of acceleration [e.g., $a^*(t) \equiv a_1^*(t)$], and with $u^*(t)$ being the corresponding component of velocity, the model for isotropic turbulence is

$$da^* = -[Ca^* + Du^*]dt + B\,dW. \quad (50)$$

This is identical to Sawford's model,[12] but with the coefficients expresses differently.

An analysis of Eq. (50) (see Refs. 12 and 13 and Sec. IV B) shows that the acceleration variance is

$$\langle a^{*2}\rangle = \frac{B^2}{2C} = a'^2\frac{\tilde{B}^2}{2\tilde{C}}, \quad (51)$$

the velocity variance is

$$\langle u^{*2}\rangle = \frac{B^2}{2CD} = u'^2\frac{\tilde{B}^2}{2\tilde{C}\tilde{D}}, \quad (52)$$

and that the Lagrangian velocity integral time scale is

$$T_L \equiv \int_0^\infty \rho(s)\,ds = \frac{C}{D} = \tau\frac{\tilde{C}}{\tilde{D}}, \quad (53)$$

where $\rho(s)$ is the Lagrangian velocity autocorrelation function defined by Eq. (2).

There is a one-to-one correspondence between the three model coefficients B, C, and D, and the three primary statistics a'^2, u'^2, and T_L. Equations (51)–(53) are readily inverted to yield for the scaled coefficients

$$\tilde{C} = \frac{T_L}{\tau}, \quad \tilde{B}^2 = \frac{2T_L}{\tau}, \quad \tilde{D} = 1. \quad (54)$$

The velocity autocorrelation function $\rho(s)$ obtained from the model is most conveniently and naturally written in terms of two different (but related) time scales, T_∞ and τ_0 $(T_\infty > \tau_0)$. These are the inverses of the two eigenvalues of the system, which are given by the solution to the quadratic equation

$$\lambda^2 - C\lambda + D = 0. \quad (55)$$

The solutions are given in terms of T_L and τ_a by

$$\lambda_1^{-1} = T_\infty = \frac{1}{2}T_L\left[1 + \left(1 - \frac{4\tau_a\tau}{T_L^2}\right)^{1/2}\right] \quad (56)$$

and

$$\lambda_2^{-1} = \tau_0 = \frac{1}{2}T_L\left[1 - \left(1 - \frac{4\tau_a\tau}{T_L^2}\right)^{1/2}\right]. \quad (57)$$

Conversely we have

$$T_L = T_\infty + \tau_0 \quad (58)$$

and

$$\tau_a = \frac{T_\infty\tau_0}{\tau}. \quad (59)$$

It may be observed that as τ_a/T_L tends to zero, T_∞ and τ_0 tend to T_L and $\tau_a(\tau/T_L)$, respectively. The coefficients B, C, and D given by Eq. (54) can be re-expressed in terms of T_∞ and τ_0, which is the form originally given by Sawford.[12]

The velocity autocorrelation function given by the model is

$$\rho(s) = \left[e^{-|s|/T_\infty} - \left(\frac{\tau_0}{T_\infty}\right)e^{-|s|/\tau_0}\right] \bigg/ \left(1 - \frac{\tau_0}{T_\infty}\right), \quad (60)$$

which is a linear combination of two decaying exponentials, with time scales τ_0 and T_∞.

To conclude, based on this examination of the model in isotropic turbulence, we summarize some important observations which are mirrored in the analysis of the general model presented below.

(1) The three model coefficients B, C, and D are uniquely related to the three primary statistics, a'^2, u'^2, and T_L.
(2) The autocorrelation function $\rho(s)$ is a linear combination of decaying exponentials, the time scales of which are the inverses of the eigenvalues of the system.
(3) The predictions of the model are in excellent agreement with Lagrangian statistics obtained from DNS (see Refs. 12 and 13).
(4) Given the primary statistics, a separate acceleration time scale cannot be imposed on the model: instead the acceleration time scale τ_a is given by Eq. (39).
(5) The simplest scaling arguments show that T_L scales with τ, and that τ_a scales with τ_η, so that the scaled coefficients $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$ are of order unity.

D. Gaussianity

For homogeneous turbulence, the model takes the form of a set of SDEs, Eqs. (31) and (32), in which the drift coefficients $(-C_{ij}a_j^*$ and $-D_{ij}u_j^*)$ are linear in the dependent variables, while the diffusion coefficient B_{ij} is independent of $\mathbf{a}^*$ and $\mathbf{u}^*$. Such linear stochastic differential equations are known[27] to yield Gaussian processes. Thus, according to the model, the processes $\mathbf{a}^*(t)$ and $\mathbf{u}^*(t)$ are jointly Gaussian.

For homogeneous turbulent shear flow, the experiments of Tavoularis and Corrsin[28] clearly show that the one-point one-time joint PDF of velocity is joint normal. Hence the model is correct in predicting that the one-time PDF $\mathbf{u}^*(t)$ is joint normal. However, it is known from DNS[11] and experiments[26,29] that both acceleration and two-time velocity statistics depart from Gaussianity, an effect which is not represented by the model. It is possible to represent these effects in stochastic models by making the model coefficients themselves stochastic processes.[30,31] In particular, Beck[31] shows that the experimental acceleration distribution can be accurately represented by a stochastic model with gamma-distributed coefficients. Here, however, we retain constant coefficients and do not attempt to represent these higher-order effects.

It is emphasized that the Gaussianity of the model is confined to homogeneous turbulence. For inhomogeneous flows, non-Gaussian statistics such as the velocity triple correlation can be accurately calculated by linear stochastic models.

E. High Reynolds number and local isotropy

We now consider the limit of very high Reynolds number, which is equivalent to the limit of τ_a/τ tending to zero. In this limit, according to the Kolmogorov hypotheses, the turbulence is locally isotropic. As is now shown, the stochastic model for acceleration is consistent with local isotropy provided that the scaled coefficients $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ tend to the following isotropic constant tensors:

$$\tilde{B}^2_{ij}=2(\tfrac{3}{4}\mathcal{C}_0)^{-1}\delta_{ij} \quad \text{and} \quad \tilde{C}_{ij}=(\tfrac{3}{4}\mathcal{C}_0)^{-1}\delta_{ij}, \tag{61}$$

where $\mathcal{C}_0$ is the Kolmogorov constant associated with the second-order Lagrangian structure function [see Eq. (69)].

In general, variations in $\mathbf{u}^*(t)$ and $\mathbf{a}^0(t)$ occur on the time scales τ and τ_a, respectively. For the case considered, $\tau_a \ll \tau$, $\mathbf{u}^*(t)$ changes very slowly compared to $\mathbf{a}^0(t)$; and so $\mathbf{a}^0(t)$ is in a statistically quasistationary state, the statistics of which change slowly in response to the changes in $\mathbf{u}^*(t)$. This state is governed by Eq. (47), with the coefficients given by Eq. (61), which can be rewritten

$$da_i^0=-(a_i^0-\mu_i[\mathbf{u}^*(t)])\frac{dt}{\frac{3}{4}\mathcal{C}_0\tau_a}+\frac{a'\,dW_i}{\sqrt{\frac{3}{8}\mathcal{C}_0\tau_a}}, \tag{62}$$

with

$$\mu_i(\mathbf{u}^*)\equiv-\frac{3}{4}\mathcal{C}_0 a'\left(\frac{\tau_a}{\tau}\right)^{1/2}\tilde{D}_{ij}\frac{u_j^*}{u'}=-\frac{3}{4}\mathcal{C}_0\frac{\tilde{D}_{ij}u_j^*}{\tau}. \tag{63}$$

With μ_i being considered as a frozen coefficient, Eq. (62) is simply the Langevin equation; and hence each component of $\mathbf{a}^0(t)$ is an independent Ornstein–Uhlenbeck (OU) process with conditional mean $\mu_i(\mathbf{u}^*(t))$, variance a'^2, and time scale $\frac{3}{4}\mathcal{C}_0\tau_a$. The normalized mean μ_i/a' tends to zero as (τ_a/τ) tends to zero [see Eq. (63)], and hence $\mathbf{a}^0(t)$ tends to a locally isotropic process.

We now examine the model equation for velocity in the high Reynolds number limit. For a general inhomogeneous flow, the model for $\mathbf{u}^*(t)$ [Eqs. (13) and (19)] is

$$\frac{du_i^*}{dt}=\left(-\frac{\partial\langle U_i\rangle}{\partial x_j}+G_{ij}^{(r)}\right)u_j^*+a_i^0(t). \tag{64}$$

As τ_a/τ tends to zero, $\mathbf{a}^0(t)$ tends to white noise; or, more precisely, for a time interval δt such that both $\tau_a/\delta t$ and $\delta t/\tau$ tend to zero, the increment in velocity

$$\int_t^{t+\delta t}\mathbf{a}^0(t')\,dt', \tag{65}$$

tends to a Gaussian random vector with mean $\boldsymbol{\mu}(\mathbf{u}^*(t))\,\delta t$, Eq. (63), and covariance

$$2a'^2(\tfrac{3}{4}\mathcal{C}_0\tau_a)\,\delta_{ij}\,\delta t=\mathcal{C}_0\epsilon\,\delta_{ij}\,\delta t. \tag{66}$$

Thus, in the limit, Eq. (64) tends to a diffusion process given by the SDE,

$$du_i^*=\left(-\frac{\partial\langle U_i\rangle}{\partial x_j}+G_{ij}\right)u_j^*\,dt+(\mathcal{C}_0\epsilon)^{1/2}\,dW_i, \tag{67}$$

with

$$G_{ij}\equiv G_{ij}^{(r)}-\frac{3}{4}\mathcal{C}_0\frac{\tilde{D}_{ij}}{\tau}. \tag{68}$$

It may be recognized that Eq. (67) is the generalized Langevin model (GLM[7,9]); and from this observation we draw two important conclusions. First, it is well known that the GLM is consistent with local isotropy and the Kolmogorov hypotheses in yielding (for the second-order Lagrangian structure function)

$$\begin{aligned}&\langle[u_i^*(t+s)-u_i^*(t)][u_j^*(t+s)-u_j^*(t)]\rangle\\&\quad=\mathcal{C}_0\epsilon s\,\delta_{ij},\quad \text{for } s\ll\tau.\end{aligned} \tag{69}$$

Second, in the high Reynolds number limit being considered, Eq. (68) gives the GLM coefficient G_{ij} which corresponds to the acceleration model coefficients $G_{ij}^{(r)}$ and $\tilde{D}_{ij}$.

For forced, statistically stationary homogeneous isotropic turbulence, the GLM coefficient G_{ij} is constrained to be $-\frac{3}{4}\mathcal{C}_0\delta_{ij}/\tau$.[9] Correspondingly, Eq. (68) yields $\tilde{D}_{ij}=\delta_{ij}$, consistent with Sawford's model, Eq. (54). In general, if the GLM coefficient G_{ij} is decomposed into slow and rapid contributions, i.e.,

$$G_{ij}=G_{ij}^{(s)}+G_{ij}^{(r)}, \tag{70}$$

then Eq. (68) yields

$$G_{ij}^{(s)}=-\frac{3}{4}\mathcal{C}_0\frac{\tilde{D}_{ij}}{\tau}. \tag{71}$$

The simplest specification of $G_{ij}^{(s)}$ (for unforced turbulence) is

$$G_{ij}^{(s)}=-\left(\frac{1}{2}+\frac{3}{4}\mathcal{C}_0\right)\frac{\delta_{ij}}{\tau}. \tag{72}$$

for which the corresponding value of $\tilde{D}_{ij}$ is

$$\tilde{D}_{ij}=\left(1+\frac{2}{3\mathcal{C}_0}\right)\delta_{ij}. \tag{73}$$

In summary, with the coefficients $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ specified by Eq. (61), the model is consistent with the Kolmogorov hypotheses. At very high Reynolds number (corresponding to τ_a/τ tending to zero), the acceleration statistics are locally isotropic, and the model tends to the generalized Langevin model (GLM) for velocity, Eq. (67). There is then a one-to-one correspondence between the remaining acceleration model coefficient $\tilde{\mathbf{D}}$ and the GLM coefficient G_{ij}, Eq. (68) and Eq. (71).

IV. HOMOGENEOUS TURBULENT SHEAR FLOW

In this section we examine the stochastic model for acceleration applied to homogeneous turbulent shear flow for which there are Lagrangian data from DNS.[16,17] The analysis (performed in Sec. IV B) depends on the processes considered being statistically stationary. We therefore define (in Sec. IV A) a scaled time $\hat{t}$, a scaled velocity $\hat{\mathbf{u}}(\hat{t})$, and the acceleration $\hat{\mathbf{a}}(\hat{t})\equiv d\hat{\mathbf{u}}(\hat{t})/d\hat{t}$ such that $\hat{\mathbf{u}}(\hat{t})$ and $\hat{\mathbf{a}}(\hat{t})$ are statistically stationary processes—at least to a reasonable approximation. Results from the analysis are compared to the DNS data in Sec. IV C.

A. Scaling for statistical stationarity

1. Forced homogeneous turbulent shear flow

We consider first the case of forced homogeneous turbulent shear flow corresponding to the DNS of Schumacher.[14] This case is relatively simple because the flow is statistically stationary. The imposed shear rate $\mathcal{S}$ is constant, as are the turbulent kinetic energy k and its dissipation rate ϵ. The nondimensional time $\hat{t}$ is defined by

$$\hat{t}\equiv t\frac{\epsilon}{k}=\frac{t}{\tau}, \tag{74}$$

and $\hat{\mathbf{u}}(\hat{t})$ is defined as the model for the fluctuating component of velocity following the fluid particle, $\mathbf{u}^*(t)$, normalized by u':

$$\hat{\mathbf{u}}(\hat{t})\equiv\frac{\mathbf{u}^*(t)}{u'}. \tag{75}$$

With these definitions, the velocity covariance $\langle\hat{u}_i\hat{u}_j\rangle$ is of order unity, and so also are the integral time scales of $\hat{\mathbf{u}}(\hat{t})$ (in scaled time). In fact, because of the equality of one-point, one-time Eulerian and Lagrangian statistics in homogeneous turbulence, we have the normalization condition following from Eq. (75):

$$\langle\hat{u}_i(\hat{t})\hat{u}_i(\hat{t})\rangle=3. \tag{76}$$

Since the velocity gradients are constant, the general stochastic model for $\mathbf{u}^*(t)$ and $\mathbf{a}^*(t)$ is given by Eqs. (31) and (32). With the transformations

$$\hat{\mathbf{u}}(\hat{t})=\frac{\mathbf{u}^*(t)}{u'},\quad \hat{\mathbf{a}}(\hat{t})\equiv\frac{d\hat{\mathbf{u}}}{d\hat{t}}=\frac{\tau\mathbf{a}^*(t)}{u'},$$
$$d\hat{t}=\frac{dt}{\tau},\quad d\hat{\mathbf{W}}(\hat{t})=\frac{d\mathbf{W}(t)}{\tau^{1/2}}, \tag{77}$$

these stochastic model equations transform to

$$d\hat{u}_i(\hat{t})=\hat{a}_i(\hat{t})d\hat{t}, \tag{78}$$

$$d\hat{a}_i(\hat{t})=-[\hat{C}_{ij}\hat{a}_j(\hat{t})+\hat{D}_{ij}\hat{u}_j(\hat{t})]d\hat{t}+\hat{B}_{ij}\,d\hat{W}_j(\hat{t}), \tag{79}$$

where the transformed (nondimensional) coefficients are

$$\hat{C}_{ij}=\tau\overline{C}_{ij}=\frac{\tau}{\tau_a}\tilde{C}_{ij}+\tau K_{ij},$$
$$\hat{D}_{ij}=\tau^2D_{ij}=\frac{\tau}{\tau_a}\tilde{D}_{ij}, \tag{80}$$

and

$$\hat{B}_{ij}=\frac{\tau^{3/2}}{u'}B_{ij}=\frac{\tau}{\tau_a}\tilde{B}_{ij}. \tag{81}$$

For a given orientation of the shear, i.e., $\partial\langle U_i\rangle/\partial x_j=\mathcal{S}\delta_{i1}\delta_{j2}$, the coefficients $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$, and $\hat{\mathbf{D}}$ are constant and depend only on the Reynolds number.

2. Unforced homogeneous turbulent shear flow

The DNS of Sawford and Yeung[16] are consistent with the supposition that (after an initial transient) the energy-containing motions in (unforced) homogeneous turbulent shear flow become (approximately) self-similar. The normalized Reynolds-stress tensor $\langle u_iu_j\rangle/k$ becomes constant, as does the ratio of the turbulence-to-shear time scales, $\tau/\tau_{\mathcal{S}}=\mathcal{S}k/\epsilon$, and hence also the ratio of production $\mathcal{P}$ to dissipation ϵ. (The values deduced from the DNS are $\mathcal{S}k/\epsilon=4.83$ and $\mathcal{P}/\epsilon=1.54$.) The turbulent kinetic energy equation then dictates that k and ϵ increase exponentially with time—as is observed.

As previously argued,[15] this picture suggests that the definitions of $\hat{t}$ and $\hat{\mathbf{u}}(\hat{t})$ by Eq. (74) and Eq. (75) remain appropriate, although now the velocity scale $u'(t)$ used in Eq. (75) depends on time. This time dependence is quantified by the parameter

$$\Pi\equiv\frac{\tau}{u'}\frac{du'}{dt}=\frac{1}{2}\left(\frac{\mathcal{P}}{\epsilon}-1\right), \tag{82}$$

the value of which is $\Pi\approx0.27$ in the present case. (The value is $\Pi=0$ for the forced case, and $\Pi=-\frac{1}{2}$ for decaying turbulence.) Given the (approximately) self-similar state of the energy-containing motions, it is reasonable to suppose that $\hat{\mathbf{u}}(\hat{t})$ is (approximately) statistically stationary. But these states can only be realized approximately since the Reynolds number increases with time. Hence, while we again define $\hat{\mathbf{a}}(\hat{t})$ as the derivative of $\hat{\mathbf{u}}(\hat{t})$, this process cannot be completely stationary: according to Kolmogorov scaling, the amplitude of $\hat{\mathbf{a}}$ increases as $R_\lambda^{1/2}$ and its time scale decreases as R_λ^{-1}.

A quantification of the variation of R_λ in homogeneous turbulent shear flow shows that the departure from stationarity is not large. Based on the exponential increase of k with time it can be shown that R_λ increases as $R_\lambda \sim \exp(\Pi \hat{t})$, and the DNS data are consistent with this behavior (except at the beginning and end of the simulation). The normalized Lagrangian velocity integral time scale is found to be $T_L/\tau = 0.3$.[15] Hence, over a time interval of $2T_L$, R_λ increases by a factor of $\exp(0.27\times 0.6)\approx 1.18$. Thus, over the relevant time interval, the amplitude of $\hat{\mathbf{a}}(\hat{t})$ increases by approximately 10%, while its time scale decreases by about 20%.

As in the forced case, the model equations [Eqs. (31) and (32)] for $\mathbf{u}^*(t)$ and $\mathbf{a}^*(t)$ can be transformed into equations for $\hat{\mathbf{u}}(\hat{t})$ and $\hat{\mathbf{a}}(\hat{t})$. The transformations are those given by Eq. (77), except that $\hat{\mathbf{a}}(\hat{t})$ is given by

$$\hat{\mathbf{a}}(\hat{t}) \equiv \frac{d\hat{\mathbf{u}}(\hat{t})}{d\hat{t}} = \tau \frac{d}{dt}\left(\frac{\mathbf{u}^*(t)}{u'(t)}\right) = \frac{\tau \mathbf{a}^*(t)}{u'} - \Pi \hat{\mathbf{u}}(\hat{t}). \tag{83}$$

The transformed model equations are again Eqs. (78) and (79), with $\hat{\mathbf{B}}$ given by Eq. (81), but with $\hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$ given by

$$\begin{aligned} \hat{C}_{ij} &= \tau \overline{C}_{ij} + 2\Pi \delta_{ij}, \\ \hat{D}_{ij} &= \tau^2 \overline{D}_{ij} + \tau \Pi \overline{C}_{ij} + \Pi^2 \delta_{ij}. \end{aligned} \tag{84}$$

It may be noted that Eq. (84) for $\hat{C}_{ij}$ and $\hat{D}_{ij}$ also applies to the forced case, since in that case Π is zero.

To conclude, the stochastic model Eq. (78) and Eq. (79) is analyzed in the next section, with the assumptions that $\hat{\mathbf{u}}(\hat{t})$ and $\hat{\mathbf{a}}(\hat{t})$ are statistically stationary. For homogeneous turbulent shear flow, the departures from stationarity are sufficiently small that the results of the analysis can usefully be compared to the DNS data of Sawford and Yeung.[17] This is done in Sec. IV C.

B. Analysis of the stochastic model

In this section we analyze the model in application to homogeneous turbulent shear flow. The analysis is somewhat involved: for the reader wishing to avoid the details, the principal results are summarized in Sec. IV B 6.

1. Model equations

When written for the scaled variables $\hat{\mathbf{u}}(\hat{t})$ and $\hat{\mathbf{a}}(\hat{t})$ in homogeneous turbulent shear flow, the model equations are Eqs. (78) and (79), and the coefficients are given by Eqs. (81) and (84).

It is convenient to use vector-matrix notation, and hence we write the model equations as

$$d\hat{\mathbf{a}}(\hat{t}) = -[\hat{\mathbf{C}}\hat{\mathbf{a}}(\hat{t}) + \hat{\mathbf{D}}\hat{\mathbf{u}}(\hat{t})]d\hat{t} + \hat{\mathbf{B}}\, d\hat{\mathbf{W}}(\hat{t}), \tag{85}$$

$$d\hat{\mathbf{u}}(\hat{t}) = \hat{\mathbf{a}}(\hat{t})d\hat{t}, \tag{86}$$

where the coefficients $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$, and $\hat{\mathbf{D}}$ are 3×3 matrices. Furthermore, it is convenient to combine $\hat{\mathbf{a}}(\hat{t})$ and $\hat{\mathbf{u}}(\hat{t})$ into a six-vector

$$\mathbf{z}(\hat{t}) \equiv \begin{bmatrix} \hat{\mathbf{a}}(\hat{t}) \\ \hat{\mathbf{u}}(\hat{t}) \end{bmatrix}, \tag{87}$$

so that the model can be written as the single SDE

$$d\mathbf{z}(\hat{t}) = -\mathbf{F}\mathbf{z}(\hat{t})d\hat{t} + \mathbf{E}\, d\overline{\mathbf{W}}(\hat{t}). \tag{88}$$

Here $\overline{\mathbf{W}}(t)$ is a six-vector-valued Wiener process, and the 6×6 matrix coefficients $\mathbf{E}$ and $\mathbf{F}$ are

$$\mathbf{E} = \begin{bmatrix} \hat{\mathbf{B}} & 0 \\ 0 & 0 \end{bmatrix} \tag{89}$$

and

$$\mathbf{F} = \begin{bmatrix} \hat{\mathbf{C}} & \hat{\mathbf{D}} \\ -\mathbf{I} & 0 \end{bmatrix}, \tag{90}$$

where $\mathbf{I}$ is the 3×3 identity matrix.

It is known from the theory of diffusion processes[9,27,32,33] that the diffusion coefficient (e.g., $\hat{\mathbf{B}}$) affects the process only through the symmetric positive–semidefinite from $\hat{\mathbf{B}}\hat{\mathbf{B}}^T$, where "$T$" denotes the transpose. Hence, without loss of generality, $\hat{\mathbf{B}}$ and therefore $\mathbf{E}$ can themselves be taken to be symmetric positive semidefinite.

It is assumed that the eigenvalues of the drift matrix $\mathbf{F}$ have positive real parts, which is a sufficient condition for Eq. (88) to yield a statistically stationary solution.[33]

2. Autocovariance

Since $\mathbf{z}(t)$ is a Gaussian process, its statistics are completely described by its autocovariance, which we define by

$$\mathbf{R}(s) \equiv \langle \mathbf{z}(\hat{t}+s)\mathbf{z}(\hat{t})^T \rangle. \tag{91}$$

It should be noted that this is the transpose of the conventional definition in that the time increment s appears in the first variable. The present definition yields simpler equations in the subsequent analysis.

The autocovariance of $\mathbf{z}(\hat{t})$ can be decomposed into the autocovariances of $\hat{\mathbf{a}}(\hat{t})$ and $\hat{\mathbf{u}}(\hat{t})$:

$$\mathbf{R}(s) = \begin{bmatrix} \mathbf{R}^{aa}(s) & \mathbf{R}^{au}(s) \\ \mathbf{R}^{ua}(s) & \mathbf{R}^{uu}(s) \end{bmatrix}, \tag{92}$$

where

$$\mathbf{R}^{ua}(s) \equiv \langle \hat{\mathbf{u}}(\hat{t}+s)\hat{\mathbf{a}}(\hat{t})^T \rangle, \tag{93}$$

and $\mathbf{R}^{aa}(s)$, $\mathbf{R}^{au}(s)$, and $\mathbf{R}^{uu}(s)$ are similarly defined.

In view of statistical stationarity, the autocovariances are independent of time $\hat{t}$ (as implied by the notation), and they possess the following symmetries:

$$\begin{aligned} &\mathbf{R}(s) = \mathbf{R}(-s)^T, \quad \mathbf{R}^{aa}(s) = \mathbf{R}^{aa}(-s)^T, \\ &\mathbf{R}^{uu}(s) = \mathbf{R}^{uu}(-s)^T, \end{aligned} \tag{94}$$

$$\mathbf{R}^{ua}(s) = \mathbf{R}^{au}(-s)^T = -\mathbf{R}^{au}(s). \tag{95}$$

Stemming from the definition $\hat{\mathbf{a}} = d\hat{\mathbf{u}}/d\hat{t}$, properties of derivatives of the autocovariances are

$$\begin{aligned} &\frac{d}{ds}\mathbf{R}^{uu}(s) = \mathbf{R}^{au}(s), \quad \frac{d}{ds}\mathbf{R}^{ua}(s) = \mathbf{R}^{aa}(s), \\ &\frac{d}{ds}\mathbf{R}^{au}(s) = -\mathbf{R}^{aa}(s), \end{aligned} \tag{96}$$

and hence

$$\frac{d^2\mathbf{R}^{uu}(s)}{ds^2} = -\mathbf{R}^{aa}(s). \tag{97}$$

Thus, all autocovariances [including $\mathbf{R}(s)$] can be determined from $\mathbf{R}^{uu}(s)$.

The covariances are denoted by

$$\mathbf{Q} \equiv \mathbf{R}(0) = \begin{bmatrix} \mathbf{Q}^{aa} & \mathbf{Q}^{au} \\ \mathbf{Q}^{ua} & \mathbf{Q}^{uu} \end{bmatrix}. \tag{98}$$

The covariance matrices $\mathbf{Q}$, $\mathbf{Q}^{aa}$, and $\mathbf{Q}^{uu}$ are symmetric positive definite; while the off-diagonal matrices are [in view of Eq. (95)] antisymmetric and the transposes of each other,

$$\mathbf{Q}^{ua} = -(\mathbf{Q}^{ua})^T = (\mathbf{Q}^{au})^T. \tag{99}$$

It may be observed from Eqs. (96) and (97) that all of the covariances can be obtained from $\mathbf{R}^{uu}(s)$ and its derivatives at the origin $(s=0)$.

An important quantity in the subsequent analysis is the *autocorrelation matrix* which is defined by

$$\mathbf{P}(s) \equiv \mathbf{R}(s)\mathbf{Q}^{-1}, \tag{100}$$

and which has the property

$$\mathbf{P}(0) = \mathbf{I}. \tag{101}$$

3. Integral time scales

The matrix $\mathbf{T}$ of integral time scales, which also plays a central role in the analysis, is defined by

$$\mathbf{T} \equiv \int_0^\infty \mathbf{P}(s)\,ds. \tag{102}$$

This matrix has a special structure, now revealed, which stems from the fact that $\hat{\mathbf{a}}(\hat{t})$ is the derivative of $\hat{\mathbf{u}}(\hat{t})$. We define the 6×6 matrix $\mathbf{M}$ by

$$\mathbf{M} \equiv \int_0^\infty \mathbf{R}(s)\,ds, \tag{103}$$

which is related to $\mathbf{T}$ by

$$\mathbf{T} = \mathbf{M}\mathbf{Q}^{-1} \quad \text{or} \quad \mathbf{M} = \mathbf{T}\mathbf{Q}, \tag{104}$$

and which is partitioned as

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}^{aa} & \mathbf{M}^{au} \\ \mathbf{M}^{ua} & \mathbf{M}^{uu} \end{bmatrix}. \tag{105}$$

For $\mathbf{M}^{aa}$ we obtain

$$\begin{aligned} \mathbf{M}^{aa} &\equiv \int_0^\infty \mathbf{R}^{aa}(s)\,ds = \int_0^\infty \langle \hat{\mathbf{a}}(\hat{t}+s)\hat{\mathbf{a}}(\hat{t})^T \rangle ds \\ &= \left\langle \int_0^\infty \hat{\mathbf{a}}(\hat{t}+s)\,ds\, \hat{\mathbf{a}}(\hat{t})^T \right\rangle \\ &= \langle [\hat{\mathbf{u}}(\infty) - \hat{\mathbf{u}}(\hat{t})]\hat{\mathbf{a}}(\hat{t})^T \rangle \\ &= -\langle \hat{\mathbf{u}}(\hat{t})\hat{\mathbf{a}}(\hat{t})^T \rangle = -\mathbf{Q}^{ua}. \end{aligned} \tag{106}$$

A similar treatment can be applied to $\mathbf{M}^{ua}$ and $\mathbf{M}^{au}$ to show that $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{bmatrix} -\mathbf{Q}^{ua} & -\mathbf{Q}^{uu} \\ \mathbf{Q}^{uu} & \mathbf{M}^{uu} \end{bmatrix}. \tag{107}$$

It then follows that $\mathbf{T}$ is of the form

$$\mathbf{T} = \begin{bmatrix} 0 & -\mathbf{I} \\ \mathbf{T}^{ua} & \mathbf{T}^{uu} \end{bmatrix}, \tag{108}$$

since the first row of the product $\mathbf{TQ}$ yields the first row of $\mathbf{M}$ [given by Eq. (107)] in accordance with Eq. (104).

By analogy to Eq. (104), we define the *velocity integral time scale tensor* by

$$\hat{\mathbf{T}}^{uu} \equiv \mathbf{M}^{uu}(\mathbf{Q}^{uu})^{-1}, \tag{109}$$

which is just (the transpose of) the time scale tensor that arises in the analysis of the stochastic model for velocity.[15] And we define the (scalar) Lagrangian velocity integral time scale by

$$T_L \equiv \tfrac{1}{3}\hat{\mathbf{T}}^{uu}_{ii}. \tag{110}$$

We see below that the autocovariance $\mathbf{R}(s)$—and therefore all other statistics—are determined by the covariance matrix $\mathbf{Q}$ and the time scale matrix $\mathbf{T}$ (as previously shown[15]). Because of the special structure of the model, the information content in $\mathbf{Q}$ and $\mathbf{T}$ is less than it appears at first sight. Specifically, the symmetric and nonsymmetric 6×6 matrices $\mathbf{Q}$ and $\mathbf{T}$ can be constructed from the 3×3 matrices $\mathbf{Q}^{aa}$, $\mathbf{Q}^{uu}$, $\mathbf{Q}^{ua}$, and $\hat{\mathbf{T}}^{uu}$—which have an information content equivalent to three symmetric and two antisymmetric 3×3 matrices. It is marvelous—although most likely inevitable—that this is precisely the information content in the model coefficients $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$, and $\hat{\mathbf{D}}$.

4. Solution for the autocorrelation matrix

It is readily deduced from the model equation, Eq. (88), that the autocovariance satisfies the ODE,

$$\frac{d}{ds}\mathbf{R}(s) = -\mathbf{F}\mathbf{R}(s) \quad \text{for } s \geq 0. \tag{111}$$

By post-multiplying both sides of this equation by $\mathbf{Q}^{-1}$, we find that $\mathbf{P}(s)$ [defined by Eq. (100)] satisfies the same equation,

$$\frac{d}{ds}\mathbf{P}(s) = -\mathbf{F}\mathbf{P}(s) \quad \text{for } s \geq 0, \tag{112}$$

with the simple initial condition $\mathbf{P}(0)=\mathbf{I}$. The solution to this equation (satisfying the initial condition) is[33]

$$\mathbf{P}(s) = \exp(-\mathbf{F}s) \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\mathbf{F}^n s^n \quad \text{for } s \geq 0, \tag{113}$$

as may be verified by differentiating with respect to s. It has been assumed that the eigenvalues of $\mathbf{F}$ have positive real part, which is a sufficient condition for $\exp(-\mathbf{F}s)$ to converge to zero as s tends to infinity.

The matrix $\mathbf{F}$ deduced from the DNS (in Sec. IV C) has the simplest structure—real positive eigenvalues, and linearly independent eigenvectors. In that case $\mathbf{F}$ can be decomposed as

$$\mathbf{F} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}, \tag{114}$$

where the columns of the 6×6 matrix $\mathbf{V}$ are the eigenvectors of $\mathbf{F}$, and $\mathbf{\Lambda}$ is the 6×6 diagonal matrix of eigenvalues. The solution for $\mathbf{P}(s)$, Eq. (113), can then be re-expressed as

$$\mathbf{P}(s)=\mathbf{V}\exp(-\mathbf{\Lambda}s)\mathbf{V}^{-1} \quad \text{for } s\geqslant 0, \tag{115}$$

showing that $\mathbf{P}(s)$ is a linear combination of six decaying exponentials, the time scales of which are the inverses of the eigenvalues.

For the general case, the time scale matrix $\mathbf{T}$ [Eq. (102)] is obtained as the definite integral of the solution, Eq. (113). The indefinite integral is

$$\int \mathbf{P}(s)ds=-\mathbf{F}^{-1}\exp(-\mathbf{F}s), \tag{116}$$

from which we obtain

$$\mathbf{T}\equiv\int_0^\infty \mathbf{P}(s)ds=\mathbf{F}^{-1}. \tag{117}$$

The 6×6 drift matrix $\mathbf{F}$ is defined in terms of the 3×3 drift matrices $\hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$ in the stochastic model for acceleration by Eq. (90). Given this structure of $\mathbf{F}$, it is readily deduced (from the equation $\mathbf{F}\mathbf{F}^{-1}=\mathbf{I}$) that its inverse is

$$\mathbf{F}^{-1}=\begin{bmatrix} 0 & -\mathbf{I} \\ \hat{\mathbf{D}}^{-1} & \hat{\mathbf{D}}^{-1}\hat{\mathbf{C}} \end{bmatrix}, \tag{118}$$

which, according to Eq. (117), equals $\mathbf{T}$. The first row of $\mathbf{F}^{-1}$ indeed matches that of $\mathbf{T}$ [Eq. (108)], while equating the elements of the second rows yields

$$\mathbf{T}^{ua}=\hat{\mathbf{D}}^{-1}, \quad \mathbf{T}^{uu}=\hat{\mathbf{D}}^{-1}\hat{\mathbf{C}}, \tag{119}$$

or conversely

$$\hat{\mathbf{D}}=(\mathbf{T}^{ua})^{-1}, \quad \hat{\mathbf{C}}=(\mathbf{T}^{ua})^{-1}\mathbf{T}^{uu}. \tag{120}$$

(The assumptions made about $\mathbf{F}$ are sufficient to ensure that $\hat{\mathbf{D}}$ is nonsingular.)

The important conclusions are that there is a one-to-one correspondence between the drift coefficients $\hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$ and the time scale matrices $\mathbf{T}^{uu}$ and $\mathbf{T}^{ua}$ and that the autocorrelation matrix $\mathbf{P}(s)$ is explicitly determined by the drift coefficients $\hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$ [through Eq. (90), Eq. (114), and Eq. (115)]. The autocovariances are given by

$$\mathbf{R}(s)=\mathbf{P}(s)\mathbf{Q}=\exp(-\mathbf{F}s)\mathbf{Q} \quad \text{for } s\geqslant 0, \tag{121}$$

where $\mathbf{F}$ is given in terms of $\hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$ by Eq. (90).

5. Solution for the covariance matrix

The solution is completed by determining the covariance matrix $\mathbf{Q}$. An evolution equation for the covariance is readily derived from the model equation [Eq. (88)], and then the condition that $\mathbf{Q}$ is independent of time yields

$$\mathbf{E}\mathbf{E}^T=\mathbf{E}^2=\mathbf{F}\mathbf{Q}+(\mathbf{F}\mathbf{Q})^T. \tag{122}$$

Thus $\mathbf{E}^2$ is twice the symmetric part of $\mathbf{FQ}$.

From the definition of $\mathbf{E}$ in terms of $\hat{\mathbf{B}}$ [Eq. (89)] we have

$$\mathbf{E}^2=\begin{bmatrix} (\mathbf{E}^2)^{aa} & (\mathbf{E}^2)^{au} \\ (\mathbf{E}^2)^{ua} & (\mathbf{E}^2)^{uu} \end{bmatrix}=\begin{bmatrix} \hat{\mathbf{B}}^2 & 0 \\ 0 & 0 \end{bmatrix}; \tag{123}$$

and from the definitions of $\mathbf{F}$ [Eq. (90)] and $\mathbf{Q}$ [Eq. (92)] we have

$$\mathbf{F}\mathbf{Q}=\begin{bmatrix} \hat{\mathbf{C}}\mathbf{Q}^{aa}+\hat{\mathbf{D}}\mathbf{Q}^{ua} & \hat{\mathbf{C}}\mathbf{Q}^{au}+\hat{\mathbf{D}}\mathbf{Q}^{uu} \\ -\mathbf{Q}^{aa} & -\mathbf{Q}^{au} \end{bmatrix}. \tag{124}$$

Equation (122) can be used to relate the blocks of $\mathbf{E}^2$ to $\mathbf{FQ}$, and evidently [from Eq. (123)] only the upper left-hand block is nonzero.

For the lower right-hand block we have, correctly,

$$(\mathbf{E}^2)^{uu}=-\mathbf{Q}^{au}-\mathbf{Q}^{auT}=0, \tag{125}$$

in view of the antisymmetry of $\mathbf{Q}^{au}$ [Eq. (99)]; and in the Appendix it is shown that the off-diagonal blocks are also zero. Thus, the only nonzero block of $\mathbf{E}^2$ given by Eq. (122) is

$$(\mathbf{E}^2)^{aa}=\hat{\mathbf{B}}^2=(\hat{\mathbf{C}}\mathbf{Q}^{aa}+\hat{\mathbf{D}}\mathbf{Q}^{ua})+(\hat{\mathbf{C}}\mathbf{Q}^{aa}+\hat{\mathbf{D}}\mathbf{Q}^{ua})^T. \tag{126}$$

6. Conclusions

The major conclusion now drawn from the analysis is that there is a one-to-one correspondence between the model coefficients ($\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$, and $\hat{\mathbf{D}}$) and the primary statistics ($\mathbf{Q}$ and $\mathbf{T}$). These primary statistics are known in terms of the velocity and acceleration covariances $\mathbf{Q}^{uu}$, $\mathbf{Q}^{ua}$, and $\mathbf{Q}^{aa}$ and the velocity integral time scale tensor $\hat{\mathbf{T}}^{uu}$, Eq. (109).

Given $\mathbf{Q}$ and $\mathbf{T}$, the coefficients $\hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$ are given by Eq. (120), and then $\hat{\mathbf{B}}^2$ is determined by Eq. (126).

Conversely, given the coefficients ($\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$, and $\hat{\mathbf{D}}$), $\mathbf{T}$ is determined by Eq. (119); and the covariances are determined by Eq. (126) together with the equation

$$\mathbf{Q}^{aa}=\hat{\mathbf{C}}\mathbf{Q}^{au}+\hat{\mathbf{D}}\mathbf{Q}^{uu}. \tag{127}$$

This equation is derived in the Appendix, where the solution of Eq. (126) and Eq. (127) for $\mathbf{Q}$ is also discussed. Together these equations yield a linear system which determines $\mathbf{Q}$, but unfortunately an explicit solution is not evident.

Once both the model coefficients and primary statistics are known, then the autocovariance given by the model $\mathbf{R}(s)=\mathbf{P}(s)\mathbf{Q}$ can be determined from Eq. (115). These autocovariances are linear combinations of the six decaying exponentials, $\exp(-\lambda_i s)$, where $\{\lambda_1,\lambda_2,\ldots,\lambda_6\}$ are the eigenvalues of the coefficient matrix $\mathbf{F}$, Eq. (90).

The analysis is complete, since the autocovariances $\mathbf{Q}(s)$ fully characterize the Gaussian model processes $\hat{\mathbf{a}}(\hat{t})$ and $\hat{\mathbf{u}}(\hat{t})$.

For Sawford's model for isotropic turbulence, the eigenvalues of $\mathbf{F}$ are τ/T_∞ and τ/τ_0 (each with multiplicity 3), corresponding to time scales T_∞ and τ_0 [Eq. (56) and Eq. (57)] which scale with the integral time scale and Kolmogorov time scale, respectively. And the time scale matrices are

$$\mathbf{T}^{uu}=\hat{\mathbf{T}}^{uu}=\left(\frac{T_\infty+\tau_0}{\tau}\right)\mathbf{I}=\frac{T_L}{\tau}\mathbf{I} \tag{128}$$

and

$$\mathbf{T}^{ua}=\frac{T_\infty\tau_0}{\tau^2}\mathbf{I}=\frac{\tau_a}{\tau}\mathbf{I}. \tag{129}$$

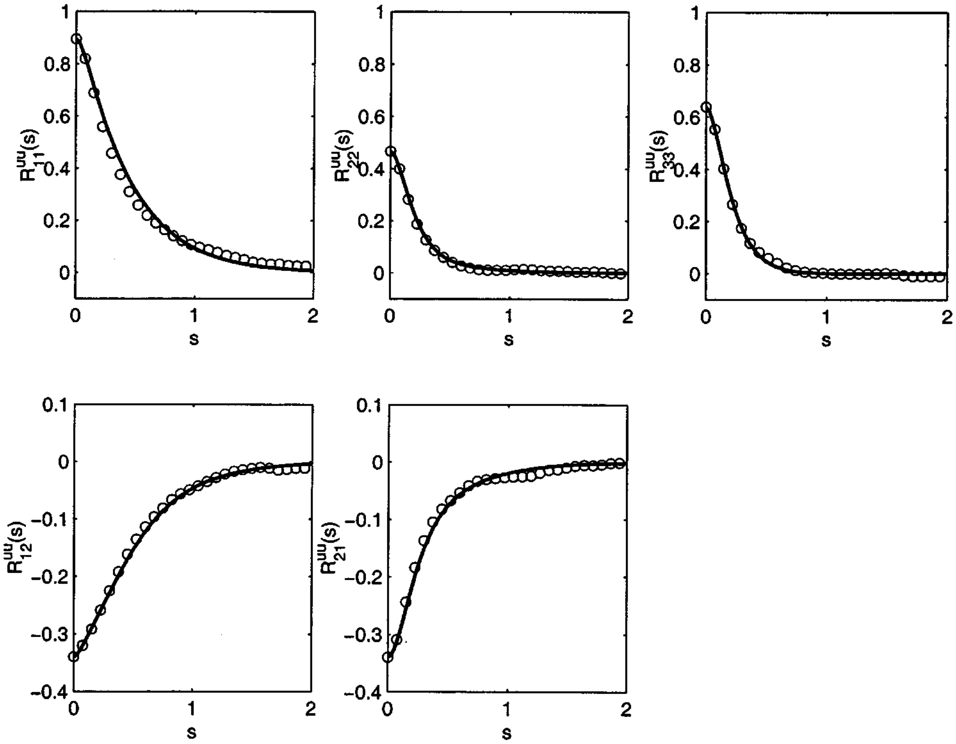

FIG. 1. Velocity autocovariances $\mathbf{R}_{ij}^{uu}(s)$ against normalized time. Symbols, DNS of Sawford and Yeung (Ref. 16); lines, from the acceleration model.

C. Comparison to DNS data

In this section, for the DNS of homogeneous turbulent shear flow,[16] the stochastic model coefficients $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$, and $\hat{\mathbf{D}}$ are deduced from the data; and then the velocity and acceleration autocovariances predicted by the model are compared to those from the DNS.

All the DNS information is extracted from the time series of the normalized velocity autocovariance $\mathbf{R}^{uu}(s)$. The remaining autocovariances [$\mathbf{R}^{ua}(s)$, $\mathbf{R}^{au}(s)$, and $\mathbf{R}^{aa}(s)$] are obtained from Eq. (96) and Eq. (97) by numerical differentiation of $\mathbf{R}^{uu}(s)$, and then the covariances are obtained as $\mathbf{Q}\equiv\mathbf{R}(0)$. Clearly this differentiation amplifies the statistical noise in the data, as is particularly evident in $\mathbf{R}^{aa}(s)$ (see Fig. 3 below). [In future DNS designed for this purpose, it would be preferable to form all covariances directly from $\hat{\mathbf{a}}(\hat{t})$ and $\hat{\mathbf{u}}(\hat{t})$.]

The velocity covariance integrals $\mathbf{M}^{uu}$ [Eqs. (103) and (105)] are formed from the time series of $\mathbf{R}^{uu}(s)$ by numerical quadrature, and then the matrix of integral time scales $\mathbf{T}$ is obtained from Eq. (104).

Based on the DNS values of the covariances and the velocity integral time scales, the values of the model coefficient $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$, and $\hat{\mathbf{D}}$ are deduced which lead to the model's matching these statistics. The values of $\hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$ are obtained from Eq. (120), and then $\hat{\mathbf{B}}^2$ from Eq. (126). (The values thus obtained are reported below.) The autocovariances predicted by the model are then deduced from Eqs. (114), (115), and (100).

Figures 1–3 show a comparison of the autocovariances from the DNS (symbols) and from the model (solid lines). Clearly the agreement is excellent, especially for the velocity autocovariances (Fig. 1). It should be noted that an acceleration time scale is not an input to the model, and so the matching of the location and magnitudes of the peaks of $\mathbf{R}^{ua}(s)$ and $\mathbf{R}^{aa}(s)$ in Figs. 2 and 3 is not inevitable.

Figure 4 compares the velocity autocovariances from the DNS, from the present acceleration model, and from the stochastic model for velocity[15] (dashed lines). Only the early times are shown where the differences between the two models are most evident. As may be seen, the acceleration model provides a much more accurate representation of the curvature of the autocovariances at small times. As the Reynolds number increases, the differences between the two models decreases, and is confined to smaller times.

The values of the coefficients $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$, and $\hat{\mathbf{D}}$ deduced from the DNS are reported in scaled form as $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{D}}$ [defined by Eq. (46)] and $\tilde{\tilde{\mathbf{C}}}\equiv\tau_a\overline{\mathbf{C}}$. Their values are

$$\tilde{\mathbf{B}}^2=\begin{bmatrix} 0.70 & -0.15 & 0 \\ -0.15 & 0.48 & 0 \\ 0 & 0 & 0.64 \end{bmatrix}, \tag{130}$$

$$\tilde{\tilde{\mathbf{C}}}=\begin{bmatrix} 0.33 & -0.03 & 0 \\ 0.07 & 0.29 & 0 \\ 0 & 0 & 0.27 \end{bmatrix}, \tag{131}$$

$$\tilde{\mathbf{D}}=\begin{bmatrix} 0.70 & 0.15 & 0 \\ 0.33 & 1.41 & 0 \\ 0 & 0 & 1.11 \end{bmatrix}. \tag{132}$$

Given that $\partial\langle U_1\rangle/\partial x_2$ is the only nonzero velocity gradient, the symmetries in the problem dictate that the off-diagonal components in the third rows and columns of these matrices are zero—as is observed. The magnitudes of all three coef-

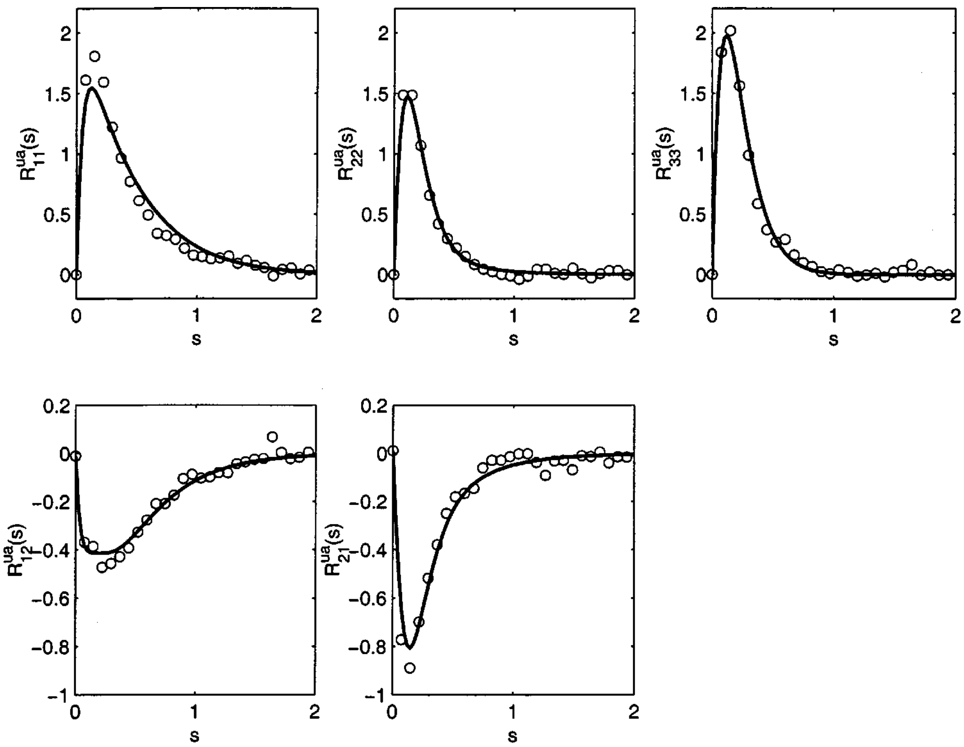

FIG. 2. Velocity-acceleration autocovariances $\mathbf{R}^{ua}_{ij}(s)$ against normalized time. Symbols, DNS of Sawford and Yeung (Ref. 16); lines, from the acceleration model. [Note that $\mathbf{R}^{au}_{ij}(s) = -\mathbf{R}^{ua}_{ij}(s)$.]

ficients are as expected from Eq. (54) given that the velocity integral time scale is $T_L/\tau \approx 0.3$.

As discussed in Sec. III E, if local isotropy prevailed at high Reynolds number, then the acceleration statistics would be isotropic (to leading order in R_λ^{-1}). A sufficient condition for the model to yield such local isotropy is that $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ (but not $\tilde{\tilde{\mathbf{C}}}$) become isotropic as the Reynolds number increases. It is evident from Eq. (130) that, at the moderate Reynolds number of the DNS, $\tilde{\underset{\sim}{\mathbf{B}}}^2$ exhibits significant anisotropy.

V. APPLICATION TO TURBULENCE MODELLING

The general model proposed here consists of an ODE Eq. (11) for the fluid-particle velocity $\mathbf{U}^*(t)$, which includes

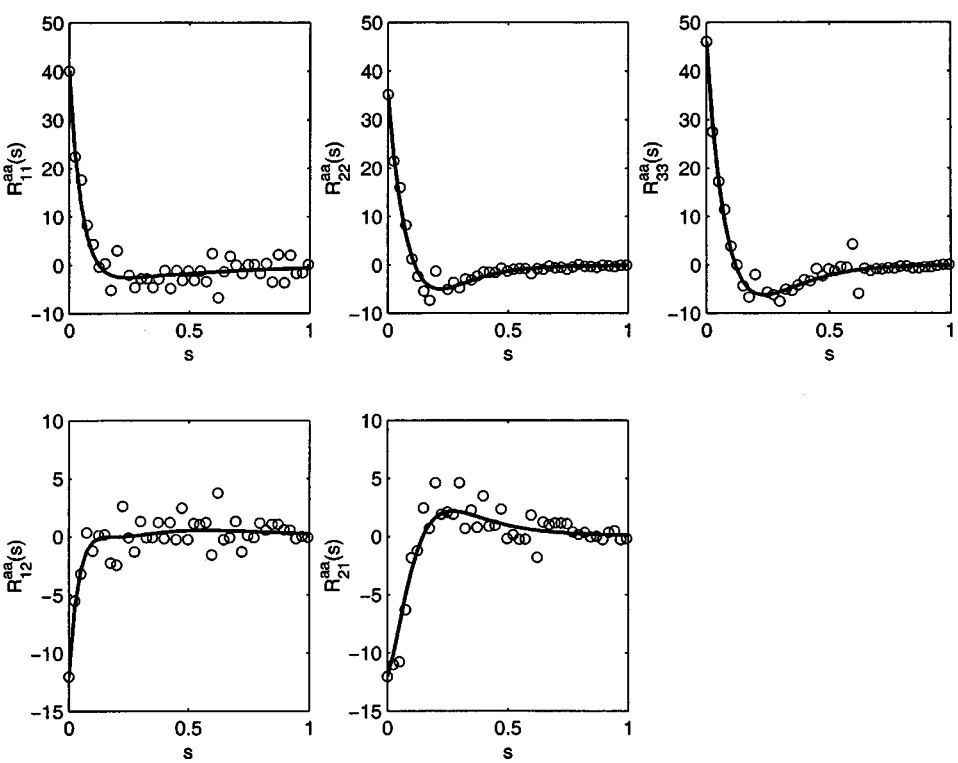

FIG. 3. Acceleration autocovariances $\mathbf{R}^{aa}_{ij}(s)$ against normalized time. Symbols, DNS of Sawford and Yeung (Ref. 16); lines, from the acceleration model.

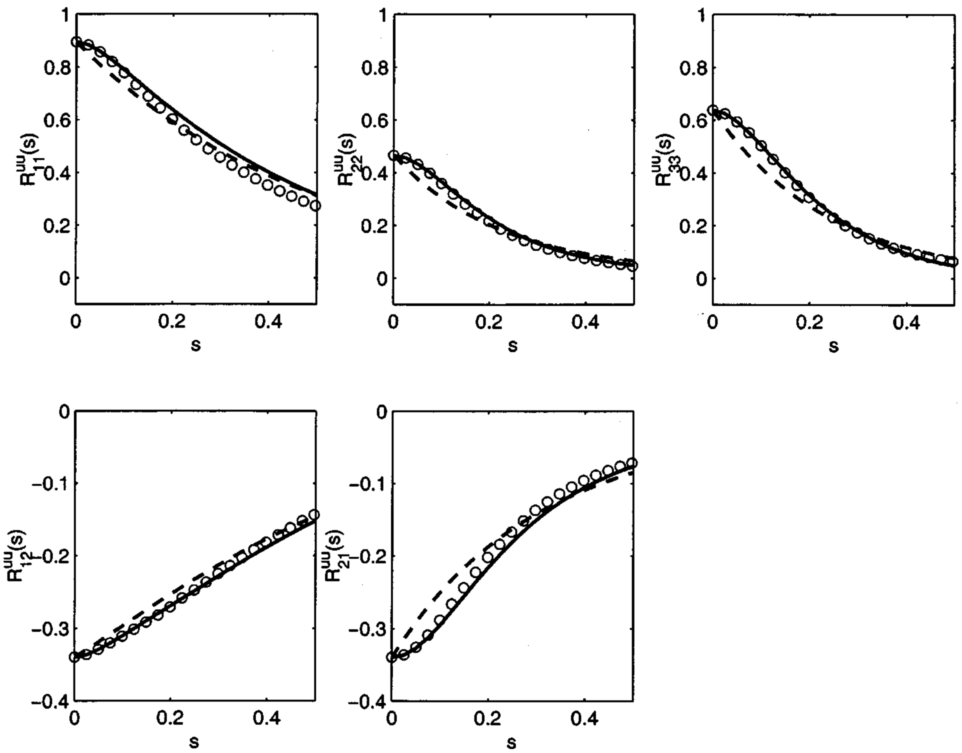

FIG. 4. Velocity autocovariances $\mathbf{R}^{uu}_{ij}(s)$ at early times. Symbols, DNS data of Sawford and Yeung (Ref. 17); solid line, from the acceleration model; dashed line, from the velocity model (Ref. 15).

a rapid-pressure model [Eq. (15)]; and an SDE [Eq. (12)] for the acceleration variable $\mathbf{A}^0(t)$. A specific model consists of a specification of the coefficients appearing in these equations, namely, N_{ijk}, B_{ij}, C_{ij}, and D_{ij}.

Beyond proposing the general model, the objective here is not to suggest a specific model, but rather to show that all of the coefficients can be deduced from DNS data on homogeneous turbulence. Hence, future DNS studies—at different Reynolds numbers and with different imposed mean velocity gradients—can be used to guide the construction of a specific model.

As outlined in the following subsections, the acceleration model can be used at different levels of turbulence modeling. In each case, the turbulent time scale τ is needed, which can be obtained from the standard model equation for ϵ or $\omega \equiv \tau^{-1}$, or from particle models for such quantities.[30,34]

A. Velocity–acceleration–wave-vector model

In addition to the model equations for $\mathbf{U}^*(t)$ and $\mathbf{A}^0(t)$, an additional SDE can be solved for the unit wave vector $\mathbf{e}^*(t)$ (Refs. 9 and 21), so that Eq. (18) can be used as the rapid-pressure model. Such a model has the virtue of representing exactly the evolution of the Reynolds stresses for arbitrary rapid distortions of homogeneous turbulence. Although it has not been convincingly demonstrated, the model should also be capable of providing a more accurate representation of the rapid pressure away from the rapid-distortion limit.

B. Velocity–acceleration model

Without the wave-vector model, the rapid pressure has to be modeled in terms of the particle velocity and Reynolds stresses (among other quantities). The standard model [Eq. (19)] is linear in the velocity, but nonlinear models [e.g., Eq. (24)] can also be considered. It has to be acknowledged that, at this level of closure, there is insufficient directional information to model accurately rapid distortions. But such models may be adequately accurate for the moderate distortions that typically occur in turbulent shear flows.

Compared to a velocity model (discussed in the next section), a velocity-acceleration model has two advantages. First, in essence it models the velocity as a second-order system, Eq. (29), rather than as a first-order system. Consequently, the rapid and slow responses of the turbulence to a sudden change in the mean velocity gradients can be modeled in a natural way. The second advantage is that acceleration is modeled realistically rather than as white noise, and thereby Reynolds number effects can be incorporated in a natural way.

An apparent disadvantage is that, in a numerical implementation, time steps Δt of order τ_a (or equivalently τ_η) are needed to resolve the acceleration time series; whereas with a velocity model the time steps can be of order T_L (or equivalently τ). With time steps of order τ_η, the computational cost increases as $\mathrm{Re}^{1/2}$. However, in most applications the details of the short-time behavior are not required, and temporal resolution on a time scale of order τ is sufficient. It is fortunate, therefore, that the model equations can be solved accurately by numerical methods that take time steps Δt that are large compared to τ_η (but small compared to τ). This is because the model coefficients vary on the time scale τ, and the model equations [e.g., Eq. (25) and Eq. (26)] with frozen coefficients admit analytic solutions. Consequently, if resolution on the Kolmogorov time scale is not required, the velocity-acceleration model can be implemented with a computational cost that is independent of Reynolds number.

C. Velocity model

Given a specific velocity-acceleration model, a corresponding velocity model can be defined—as now outlined.

When applied to (approximately) self-similar homogeneous turbulence (at a given Reynolds number and with given imposed mean velocity gradients), the velocity-acceleration model yields a value of the normalized Reynolds stress tensor ($\mathbf{Q}^{uu}$) and of the velocity integral time scale tensor ($\hat{\mathbf{T}}^{uu}$). The corresponding velocity model is defined to be the linear SDE for velocity that yields these same statistics. The drift and diffusion coefficients in the velocity model are uniquely determined by $\mathbf{Q}^{uu}$ and $\hat{\mathbf{T}}^{uu}$.[15]

[This procedure for determining the velocity-model coefficients is straightforward to implement numerically; but an analytical treatment is hampered by the lack of an explicit solution to Eq. (122) for $\mathbf{Q}$.]

The form of the velocity model thus obtained is the same as the generalized Langevin model (GLM[7,9]) but with an anisotropic diffusion coefficient. The advantage of obtaining a velocity model by this route is that it inherits the Reynolds-number dependence (and other attributes) of the velocity-acceleration model. At very high Reynolds number the acceleration model tends to the GLM with isotropic diffusion, Eq. (67), and with the coefficient G_{ij} given by Eq. (71).

D. Reynolds-stress model

Given a particle model for velocity, it is straightforward to derive a corresponding Reynolds-stress equation.[10,20,35] Again, such a model inherits from its antecedents a Reynolds-number dependence and other attributes.

VI. CONCLUSIONS

We have considered a stochastic model for fluid-particle velocity and acceleration in inhomogeneous turbulent flows. The model consists of an ODE for velocity, Eq. (11), and an SDE for an acceleration variable, Eq. (12). This structure produces the correct qualitative response to rapid distortions. If the model is supplemented by the wavevector equation, then the resulting model [Eq. (15) and Eq. (18)] is exact for arbitrary rapid distortions of homogeneous turbulence. Otherwise, a standard linear model, Eq. (19), for the rapid pressure can be used.

For isotropic turbulence, the SDE for acceleration reduces to Sawford's model.[12] For very high Reynolds number the model is consistent with local isotropy and the Kolmogorov hypotheses, and tends to the generalized Langevin model for velocity. For homogeneous turbulence (with constant and uniform imposed mean velocity gradients) a full analysis of the model is performed. This establishes the one-to-one correspondence between the model coefficient tensors **B**, **C**, and **D** and the primary statistics of the model, namely, the velocity-acceleration covariances and the velocity integral time scale tensor. Details are given in Sec. IV B 6. For homogeneous turbulence, the modeled processes (i.e., the velocity and acceleration time series) are Gaussian, and hence are completely characterized by their autocovariance, which is given explicitly by Eq. (121). The Gaussianity of acceleration and of multitime velocity statistics is physically incorrect, and reflects the fact that the model does not account for internal intermittency.

For homogeneous turbulent shear flow, the model coefficients are evaluated from the DNS data of Sawford and Yeung.[17] The model autocovariances thus obtained (Figs. 1–4) are in excellent agreement with those from the DNS, including the short-time (Kolmogorov scale) behavior (see Fig. 4).

Compared to a linear stochastic model for velocity, the velocity-acceleration model has the advantage of providing a realistic representation of the behavior on the Kolmogorov time scale; and, as a consequence, of naturally incorporating Reynolds-number effects. The purpose here has not been to propose a specific model (i.e., a specification of the model coefficients), but rather to show that these coefficients can be deduced from DNS of homogeneous turbulence, as functions of the Reynolds number and of the imposed mean velocity gradients.

As discussed in Sec. V, the velocity-acceleration model can be used as a basis for generating a range of (Reynolds-number dependent) PDF and Reynolds-stress turbulence models.

ACKNOWLEDGMENTS

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APPENDIX: SOLUTION FOR THE COVARIANCE MATRIX

The purposes of this appendix are to derive Eq. (127); to show that the off-diagonal blocks of $\mathbf{E}^2$ given by Eq. (122) and Eq. (124) are zero; and to discuss the solution of Eq. (126) and Eq. (127) for the covariances.

The covariances are related to the derivatives of the autocovariances at the origin, Eq. (96). From the ODE for the model autocovariance [Eq. (111)] we obtain

$$\left[\frac{d}{ds}\mathbf{R}(s)\right]_{s=0} = -\mathbf{F}\mathbf{Q} = -\begin{bmatrix} \hat{\mathbf{C}} & \hat{\mathbf{D}} \\ -\mathbf{I} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{Q}^{aa} & \mathbf{Q}^{au} \\ \mathbf{Q}^{ua} & \mathbf{Q}^{uu} \end{bmatrix} = \begin{bmatrix} -\hat{\mathbf{C}}\mathbf{Q}^{aa} - \hat{\mathbf{D}}\mathbf{Q}^{ua} & -\hat{\mathbf{C}}\mathbf{Q}^{au} - \hat{\mathbf{D}}\mathbf{Q}^{uu} \\ \mathbf{Q}^{aa} & \mathbf{Q}^{au} \end{bmatrix}. \tag{A1}$$

The bottom row of this last matrix is consistent with the first two relations in Eq. (96); while the consistency of the upper right block with the third relation in Eq. (96) yields

$$\mathbf{Q}^{aa} = \hat{\mathbf{C}}\mathbf{Q}^{au} + \hat{\mathbf{D}}\mathbf{Q}^{uu}, \tag{A2}$$

which is Eq. (127).

The second derivative at the origin is

$$\left[\frac{d^2}{ds^2}\mathbf{R}(s)\right]_{s=0} = \mathbf{F}^2\mathbf{Q}. \tag{A3}$$

By expanding the right-hand side and invoking Eq. (97) we obtain

$$-\left[\frac{d^2}{ds^2}\mathbf{R}^{uu}(s)\right]_{s=0}=\mathbf{Q}^{aa}=\hat{\mathbf{C}}\mathbf{Q}^{au}+\hat{\mathbf{D}}\mathbf{Q}^{uu}, \tag{A4}$$

which provides no new information, but is consistent with Eq. (A2).

From Eq. (122) and Eq. (124), we obtain for the upper right block of $\mathbf{E}^2$,

$$(\mathbf{E}^2)^{au}=\hat{\mathbf{C}}\mathbf{Q}^{au}+\hat{\mathbf{D}}\mathbf{Q}^{uu}-\mathbf{Q}^{aa^T} \tag{A5}$$

$$=\mathbf{Q}^{aa}-\mathbf{Q}^{aa^T}=0, \tag{A6}$$

where the second line follows from Eq. (A2) and the symmetry of $\mathbf{Q}^{aa}$.

We turn now to the solution of Eq. (126) and Eq. (127) for $\mathbf{Q}^{uu}$, $\mathbf{Q}^{aa}$, and $\mathbf{Q}^{ua}$ given $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$, and $\hat{\mathbf{D}}$. Recall that $\mathbf{Q}^{uu}$ and $\mathbf{Q}^{aa}$ are symmetric, while $\mathbf{Q}^{ua}$ is antisymmetric. Both sides of Eq. (126) are identically symmetric, whereas the right hand side of Eq. (127) is not identically symmetric. Hence, together, these equations represent a linear system for the components of $\mathbf{Q}^{uu}$, $\mathbf{Q}^{aa}$, $\mathbf{Q}^{ua}$—with the same number of independent equations as the number of independent unknowns, i.e., 15.

It is very unfortunate that there appears not to be a simple explicit solution for the covariances. It should be possible, however, to obtain an explicit solution using tensor representation theorems.[36,37] That is, the covariance can be written

$$\mathbf{Q}^{uu}=\sum_{n=1}^{N_s} r_{uu}^{(n)}\mathbf{S}^{(n)}, \quad \mathbf{Q}^{aa}=\sum_{n=1}^{N_s} r_{aa}^{(n)}\mathbf{S}^{(n)},$$
$$\mathbf{Q}^{ua}=\sum_{n=1}^{N_a} r_{ua}^{(n)}\mathbf{A}^{(n)}, \tag{A7}$$

where $\{\mathbf{S}^{(n)}\}$ is a complete set of N_s linearly independent symmetric tensor functions that can be formed for $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$, and $\hat{\mathbf{D}}$; and similarly $\{\mathbf{A}^{(n)}\}$ is a complete set of N_a antisymmetric tensors. The coefficients $\{r_{uu}^{(n)}\}$, $\{r_{aa}^{(n)}\}$, and $\{r_{ua}^{(n)}\}$ can then be deduced from Eq. (126) and Eq. (127): they are invariants of $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$, and $\hat{\mathbf{D}}$. However such a solution is unlikely to be simple (or easy to obtain).

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