A PERSPECTIVE ON TURBULENCE MODELING

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1. Introduction

In this paper, we speculate on the usage of turbulence models over the next 10 or 20 years. To this end, some insights are gained by reflecting on the developments over the last 25 years, since the introduction of the k-ε turbulence model (Jones *et al.*, 1972).

It is emphasized (in Section 2) that there currently exist a broad range of models and a broader range of applications. In 10 or 20 years, computer power will have advanced substantially. Will this lead to an abandonment of the simpler models (based on the Reynolds-averaged Navier-Stokes (RANS) equations), in favor of large-eddy simulations (LES)? The consequences of increased computer power are considered in Section 3. It is argued that, in most applications, there are more pressing demands for the increased computer power than the use of computationally-intensive approaches such as LES. Consequently, while the use of LES will undoubtedly increase, it is argued that the bulk of turbulent flow calculations will still be based on RANS (or unsteady RANS) approaches. The whole range of turbulence models will remain useful.

The general opinion (e.g., Bradshaw *et al.* (1996)) is that the progress towards accurate and general RANS models (Reynolds-stress models (RSM) in particular) has been disappointingly slow. As discussed in Section 4, it is important to distinguish between the *level of closure* and *particular models*. The level of closure defines a class of models; and a particular model is a member of one such class. If, in some application, a particular turbulence model is found to be inaccurate, is this because the turbulence physics cannot be adequately represented at the level of closure? Or is it because the coefficients in the particular model have been poorly chosen? While every level of closure has its intrinsic limitations, it is suggested that the some-

what disappointing performance of RANS models is due more to the choice of model coefficients. Specifically, current models are some distance from the *optimal model* at that level of closure.

Optimal models are defined and discussed in Section 5. Since models at different levels of closure will be used for many years to come, it would be extremely valuable to develop a general methodology to determine optimal models. (It is remarkable how little effort over the last 25 years has been applied to this objective.)

2. Current Range of Models and Applications

Nearly all modeling approaches in use and envisioned fall into one of the following three classes:

1. RANS models.
2. Partial resolution of unsteady turbulent motions.
3. Full resolution of unsteady turbulent motions.

RANS models can be applied to statistically stationary flows, for which they predict the stationary fields of the mean velocity and some turbulence properties. These models include:

- algebraic models (e.g., Baldwin *et al.* (1978))
- one-equation models (e.g., Spalart *et al.* (1992))
- two-equation models (e.g., Jones *et al.* (1972))
- Reynolds-stress models (e.g., Launder *et al.* (1975))
- elliptic relaxation models (e.g., Durbin (1993))
- PDF models (e.g., Haworth *et al.* (1986a)).

Even though they lack generality, it is important to appreciate that the simplest models can perform quite satisfactorily for narrow—but technologically important—classes for flows (see, e.g., Wilcox (1993)).

RANS models represent the turbulence in terms of one-point one-time statistics, and they do not explicitly represent time-dependent turbulent motions. In some circumstances—when there are large-scale unsteady turbulent motions that have a dominant effect upon the flow—it may be preferable to account explicitly for some of these unsteady motions. One way is through large-eddy simulation: but it should be appreciated that there is again a range of possibilities for the *partial resolution of unsteady turbulent motions*. Such possibilities include:

- unsteady RANS modeling—in which only the largest (usually 2D) unsteady motions are resolved
- very large eddy simulations (VLES)—in which some fraction of the turbulent energy-containing motions are resolved

– LES—in which essentially all of the energy-containing motions are resolved.

Durbin (1995) provides an excellent example of the success of unsteady RANS for a flow with vortex shedding, for which the corresponding steady RANS model calculations are grossly inaccurate. The computational cost and difficulty of such unsteady RANS calculations is orders of magnitude less than LES calculations.

In LES (and other partial resolution approaches) modeling is required. Only by fully resolving all scales of turbulent motion through direct numerical simulation (DNS) can all modeling be avoided.

DNS is a powerful research tool that the author has used extensively (e.g., Yeung et al. (1989); Juneja et al. (1996).) But DNS should not be considered as a possible approach to engineering turbulent flow calculation (at moderate or high Reynolds number, within the twenty-year time frame being considered here). Not only is the computational cost outrageous and prohibitive, but the computational effort is misplaced. From an engineering perspective, in nearly all circumstances it is the energy-containing turbulent motions that are important, whereas in DNS over 99.9% of the effort is devoted to the smallest dissipative motions.

To substantiate this observation, consider a DNS of homogeneous isotropic turbulence using a pseudo-spectral method. Spatial resolution requires the highest wavenumber k_{max} to satisfy

$$k_{max}\eta \geq 1.5,$$

where η is the Kolmogorov lengthscale. The dissipative range can be taken to be wavenumbers greater than k_D, where

$$k_D\eta = 0.1.$$

In the three-dimensional wavenumber space, the fraction of the wavenumbers that are in the dissipative range is

$$\frac{k_{max}^3 - k_D^3}{k_{max}^3} \geq \frac{15^3 - 1}{15^3} = 0.9997.$$

Thus, 99.97% of the computational effort is devoted to the dissipative motions.

Not only is there a broad range of turbulence models, but there is also a broad range of uses to which they are put. The following are some of the distinguishing characteristics of the different uses:

– geometric complexity. From simple 2D airfoils, to the flow in the ports and cylinders of internal combustion engines.

- complexity of physical processes. Applications consisting solely of single-phase fluid mechanics are the exception rather than the rule. More often, the applications involve heat or mass transfer, chemical reactions, multi-phase flow etc.
- accuracy required. In many aerodynamic applications, calculations are required to be accurate to less than 1%. In more complex, less well understood flows, 20% accuracy may be very useful.
- number of calculations. In some applications, considerable effort can be justified to make a calculation of a single flow. In others, such as design optimization, possibly thousands of flow calculations are required.
- user's background and support. In some industrial settings, turbulence-model calculations are performed by highly-qualified (e.g., Ph.D. degree) and experienced personnel; and there may be a parallel experimental program to validate the models for the range of flows considered. Increasingly though, commercial CFD codes incorporating turbulence models are being used by less qualified (e.g., M.S. degree) and less experienced personnel, with less experimental validation.
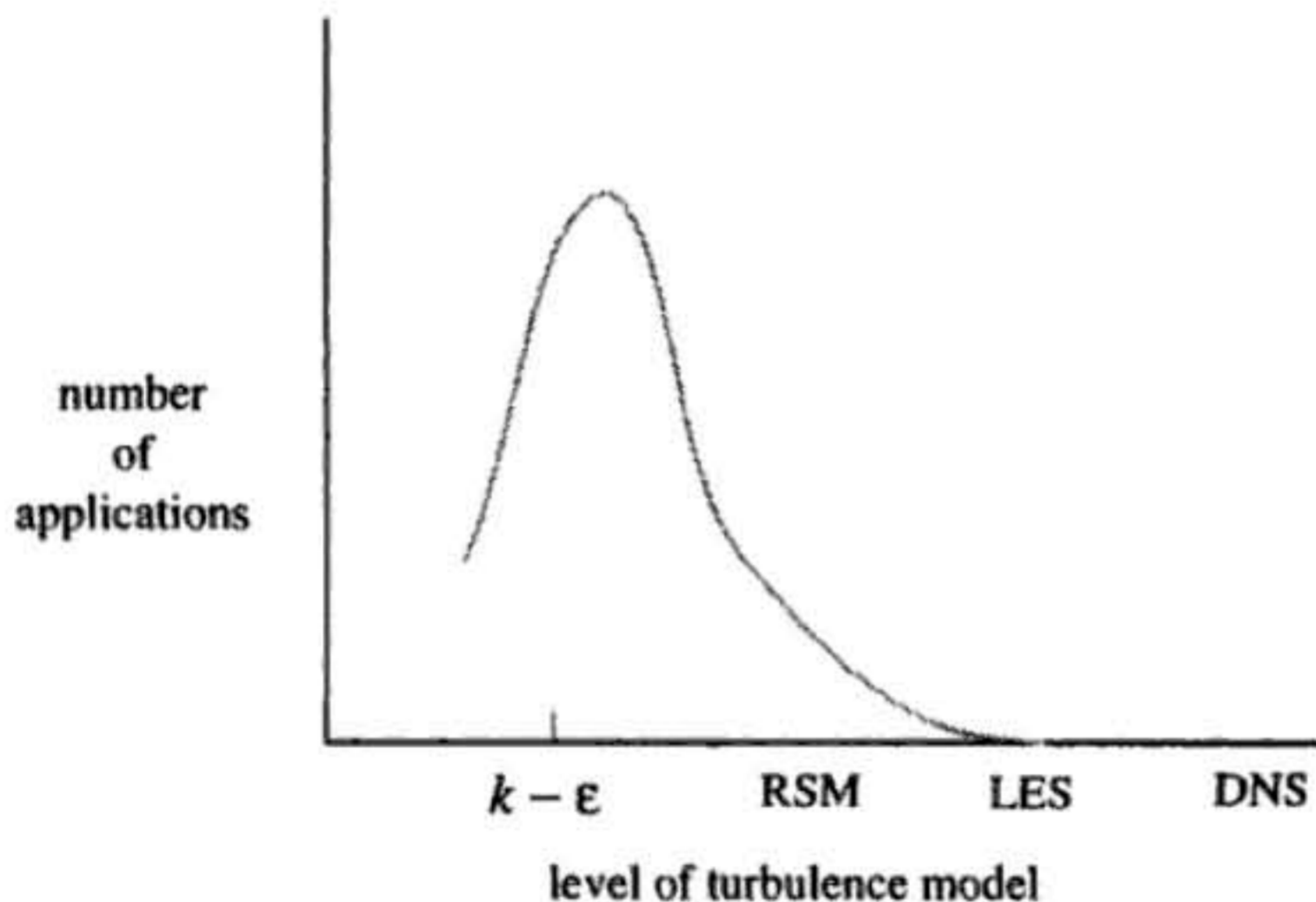

Figure 1. Distribution of current turbulence model usage.

Figure 1 provides a rough picture of current turbulence model usage in engineering applications. The center of the distribution is squarely at the two-equation level (e.g., k-ε): this is the simplest level at which a *complete closure* is possible (without requiring the specification of a mixing length, for example); and the k-ε model is implemented in nearly all commercial CFD codes. But the range extends to yet simpler models (one-equation

and algebraic) and also to Reynolds stress models, with a handful of LES applications.

3. Impact of Increasing Computer Power

Figure 2 shows the speed (measured in flops—floating point operations—per second) of the fastest computers over the last 30 years. It may be seen that this speed has increased quite consistently at a rate of a factor of 30 per decade. While there is no sound basis for extrapolation beyond a few years, it is nevertheless generally supposed that this trend will continue.
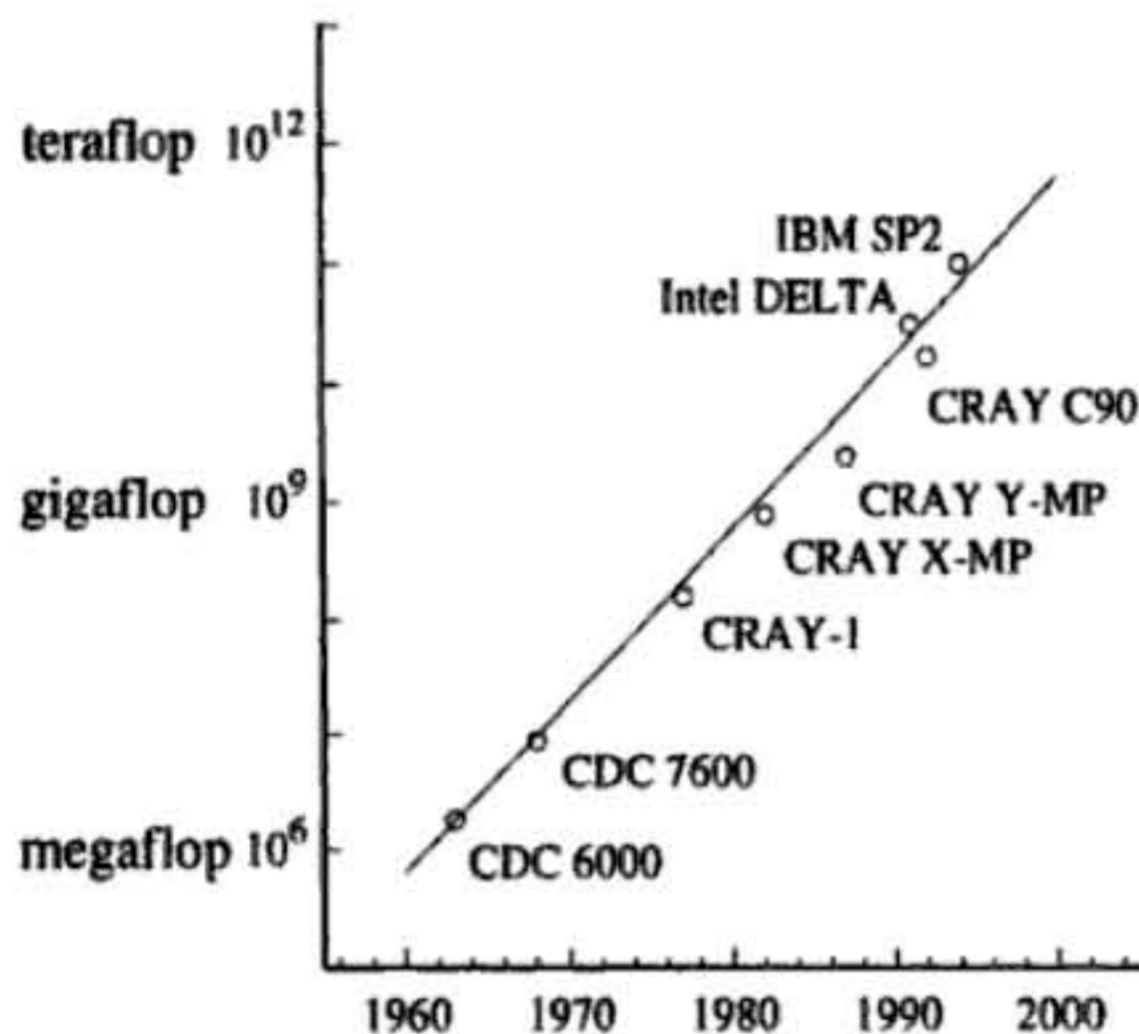

Figure 2.   Speed (flops per second) of the fastest supercomputers against year of their introduction. The line shows a growth rate of a factor of 30 per decade. (Adapted from Foster (1995).)

Clearly, an increase in computer power of 30 in 10 years or 900 in 20 years will have a major impact on how turbulent flow calculations are performed. But the nature of this impact requires careful consideration.

3.1. RANGE OF COMPUTATIONS

There is, of course, a large range in the scale of turbulent flow computations performed. It is useful to characterize this range by three types of calculations.

1. large-scale research computations requiring of order 200 CPU hours on the most powerful supercomputer. The channel-flow DNS of Kim

(1987) required 250 hours. Such calculations are generally much too expensive for engineering applications.

2. large engineering calculations requiring 15 minutes CPU time on a supercomputer, or equivalently 25 hours CPU time on a workstation.

3. repetitive engineering calculations—as may be required in a design optimization study—requiring 1 minute of CPU time on a workstation.

The relative magnitudes of these computations is summarized in Table 1. For use in a "large engineering calculation" a methodology that is currently a "large-scale research computation" requires an increase in computer speed of about 1,000. With a speed increase rate of 30 per decade, this corresponds to 20 years. To use the same methodology for "repetitive engineering computations" requires a further factor of 1,000 in speed, and hence a further 20 years of hardware development. (These estimates of 20 and 40 years may be pessimistic by a factor of 2, say, because they do not take into account algorithmic developments, which typically keep pace with hardware developments.)

TABLE 1. CPU time requirements of different types of computations.

	large-scale research computation	large engineering calculation	repetitive engineering computation
CPU time on supercomputer	200 hrs	15 min	—
CPU time on workstation	—	25 hrs	1 min
relative CPU time	1.2×10^6	1.5×10^3	1
	now	20 years	40 years
	—	now	20 years

These considerations suggest that relatively simple models, that currently require significantly less CPU time than "large-scale research computations," will continue to be dominant in engineering usage over the next 20 years.

3.2. USE OF INCREASED COMPUTER POWER

In considerations of turbulent flow calculations, it is often assumed that all of the future increases in computer power can be used for more demanding turbulence approaches—LES in place of RANS, for example. In fact, there are many demands on the computer resources, and turbulence modeling may be far from the most pressing. Examples of the uses of increased computer power are:

1. improved spatial resolution in the computations to achieve numerical accuracy
2. more complete and accurate representation of the boundary geometry
3. use of larger solution domains so that boundary conditions can be specified more easily
4. modeling of other processes such as: heat and mass transfer; chemical reactions; multi-phase flow; acoustics
5. use of more versatile, general and user-friendly codes (which are less efficient in terms of CPU time)
6. reduce run times—to enable repetitive engineering calculations, for example
7. use of more demanding turbulence models.

3.3. DETERMINANTS OF COMPUTER REQUIREMENTS

In order of importance, the primary factors determining the computer requirements of a turbulence modeling approach are:

1. length and time scales that have to be resolved
2. number (and nature) of independent variables
3. number (and nature) of equations to be solved.

DNS requires the resolution of the smallest Kolmogorov scales, which is why it is prohibitively demanding for moderate and high Reynolds number flows. RANS approaches, on the other hand, require resolution only of mean fields. In approaches with partial resolution of the unsteady turbulent motions, the computer requirements rise rapidly as the scales of the resolved motions decreases. For statistically-stationary flows, statistical approaches (e.g., RANS) have no temporal resolution requirements, and converged solutions can be obtained in of order 100 iterations. In contrast, for LES and DNS time-marching methods are required, and typically tens of thousands of time steps are needed.

Not all independent variables are the same. The computer requirements increase rapidly going from OD to 1D to 2D and to 3D flows. But in PDF methods, the computer work rises only linearly (to a first approximation) with the number of independent sample-space variables. For example, Saxena et al. (1998) performed PDF calculations with 18 independent variables (2 positions, 3 velocities, 1 turbulence frequency, and 14 thermochemical compositions) for a piloted methane jet flame. This required just 25 CPU hours on a workstation.

In comparison with items 1 and 2 in the list above, the impact of item 3—the number of equations solved—is relatively minor.

3.4. COMPUTATIONAL COSTS

We have discussed CPU times extensively above. But this is just one component of the cost of performing a turbulent flow calculation. The primary components of the cost are

1. code development costs (paid directly by the developer, or indirectly by the user through license fees)
2. manpower costs in performing the calculations (salary, education and training)
3. hardware costs (CPU time, etc.).

At commercial rates, large DNS calculation can cost well over $1,000,000, and so for this case item 3 dominates. And in such applications, computational efficiency (in terms of CPU time) is of prime importance.

With simpler models, the other components of cost may be more significant. Indeed, as illustrated in Fig. 3, as CPU speeds increase, eventually the CPU costs become small compared to the other costs. In these circumstances, rather than CPU efficiency, the important issues become ease of use, reliability and accuracy.
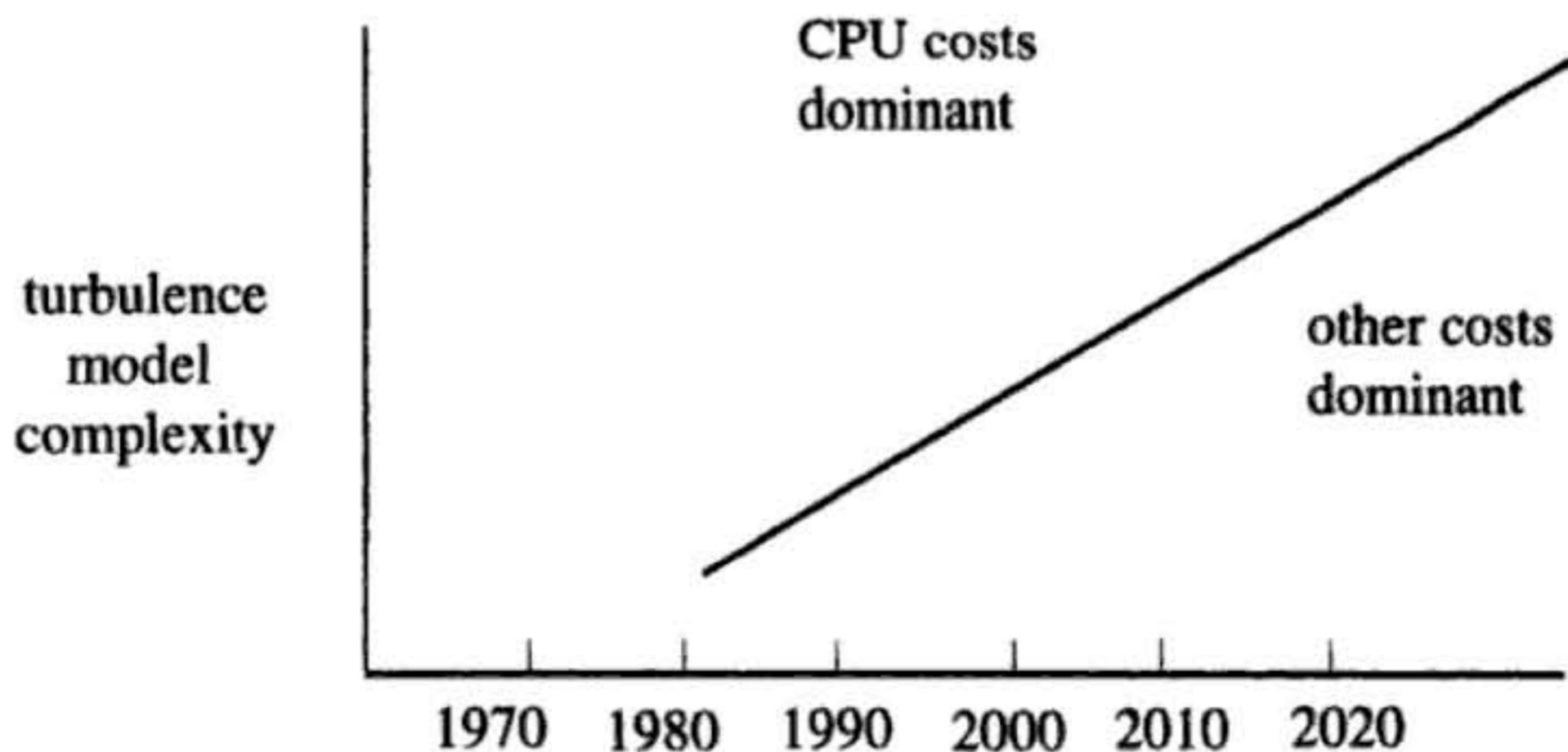
Figure 3. Relative importance of computation costs for turbulence models of different complexities. With increasing time, as CPU speeds increase, CPU costs become less dominant.

4. Future Usage of Turbulence Models

Figure 4 shows the conjectured distribution of turbulence model use 10 or 20 years from now. The main points are

1. The total usage will increase

    (a) because of the increased availability and ease-of-use of commercial CFD codes

    (b) because the increased computer power allows the treatment of more applications (most with complex geometry and complex physical processes)

    (c) because of model improvements, the accuracy (and hence usefulness) of the calculations will increase

2. Most of the increases in computer power will be used for items 1–6 enumerated in Section 3.2, not for more complex turbulence modeling. Hence the distribution shown in Fig. 4 moves only slightly in the direction of more demanding models.

3. There will be more use of models which partially resolve the unsteady turbulent motions (e.g., unsteady RANS and LES), but RANS methods remain dominant.
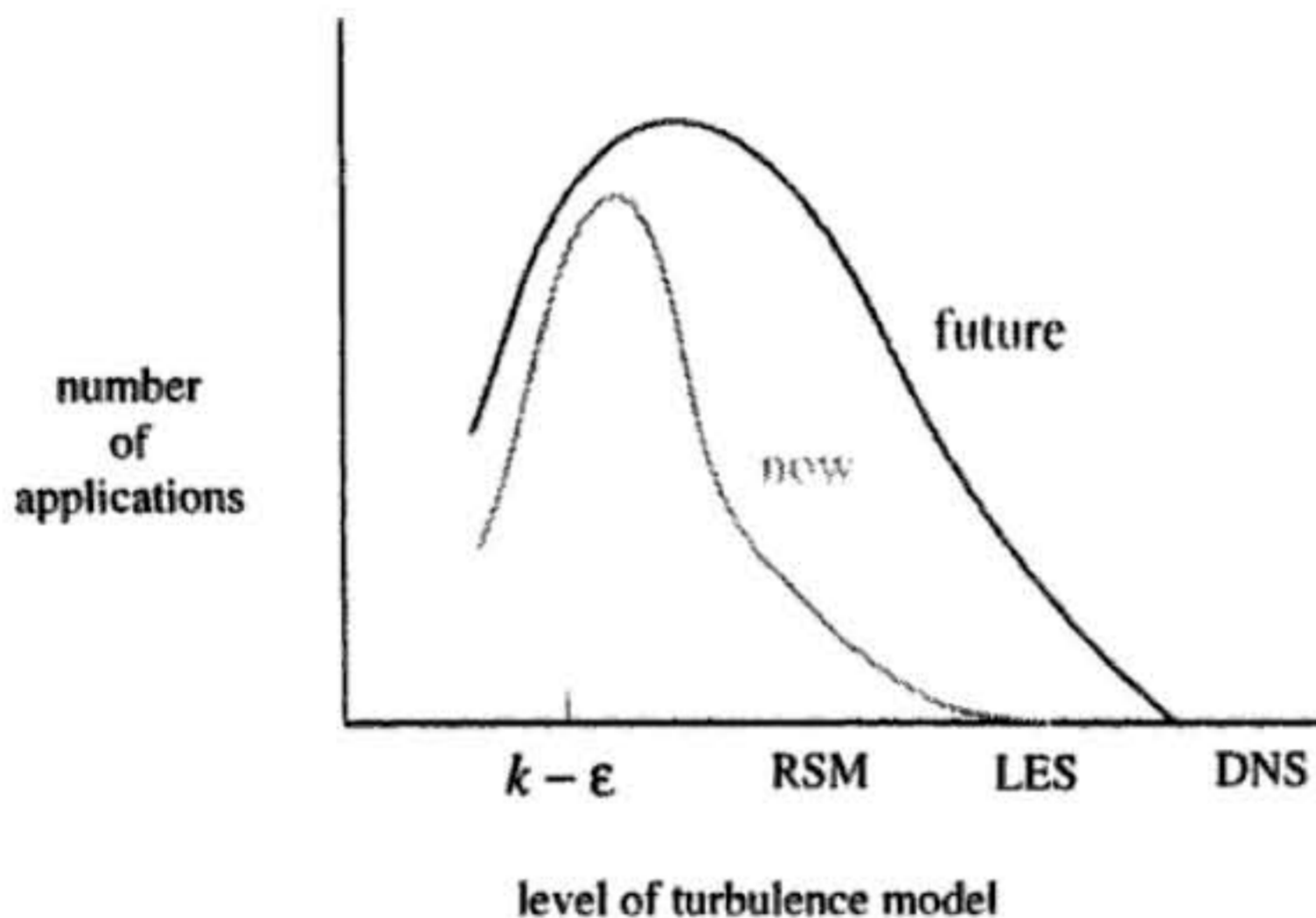

Figure 4. Distribution of current and future (10–20 years) turbulence model usage.

Further to (3) above, we can ask: how will the increased usage in engineering of "partial resolution" models take place? And, in this context, what developments in modeling methodology will be useful? On the first question, one route that may be followed in some applications is the abandonment of current RANS-based methods, with a jump to LES. More prevalent, however, in the industrial setting will be a more gradual transition from

RANS, to unsteady RANS, to VLES, etc. At present, there are distinct model equations solved at the different levels of resolution (RANS–LES–DNS). A useful aim of modeling research in this area is to develop a unified model, which can be applied (and is accurate) at any specified level of resolution, from RANS to DNS.

As discussed in Section 2, there is a broad range of turbulent flow applications, most involving more complex physical and chemical processes than the constant-property Newtonian flows that form the focus of current LES research. The development of "partial resolution models" for such complex processes is necessary if this methodology is to be broadly applied. As an example, Colucci *et al.* (1998) extend the PDF methodology to treat chemically reactive flows by LES.

5. Levels of Closure and Particular Models

Based on a collaborative testing of turbulence models, Bradshaw *et al.* (1996) summarize the status of RANS modeling thus: "Our conclusion is, alas, much the same as that of the 1980-81 meeting: no current Reynolds-averaged turbulence model can predict the whole range of complex turbulent flows to worthwhile engineering accuracy." "Stress-transport models ... did appear somewhat better than eddy-viscosity methods, but not enough to warrant the abandonment of eddy-viscosity models." This disappointing state of affairs prompts a number of questions, primarily:

- why has more progress not been made in the last 25 years?
- have RANS models reached their full potential, or are there RANS models (yet to be discovered) with markedly superior performance?

As a first step in a discussion of these questions we emphasize the distinction between the *level of closure* (or *class of models*) and *particular models*.

Figure 5 shows an incomplete classification of RANS models. The models are classified (and sub-classified) in terms of the *forms* of their constituent equations. A particular model—a point in Fig. 5—is characterized by the complete specification of all of the coefficients appearing in the model equations.

Consider, for example, the class (not the most general) of k-ε models defined by the equation:

$$\langle u_i u_j \rangle = k \mathcal{F}_{ij}(\widehat{\mathbf{S}}, \widehat{\mathbf{\Omega}}), \tag{1}$$

$$\frac{\bar{D}k}{\bar{D}t} = \nabla \cdot \left(\frac{C_\mu k^2}{\sigma_k \varepsilon} \nabla k \right) - \langle u_i u_j \rangle \frac{\partial \langle U_i \rangle}{\partial x_j} - \varepsilon, \tag{2}$$

$$\frac{\bar{D}\varepsilon}{\bar{D}t} = \nabla \cdot \left(\frac{C_\mu}{\sigma_\varepsilon} \frac{k^2}{\varepsilon} \nabla \varepsilon \right) + \frac{\varepsilon^2}{k} S_\varepsilon(\widehat{\mathbf{S}}, \widehat{\mathbf{\Omega}}), \tag{3}$$

where $\langle \mathbf{U} \rangle$ is the mean velocity, $\bar{D}/\bar{D}t$ is the mean substantial derivative $(\partial/\partial t + \langle \mathbf{U} \rangle \cdot \nabla)$ and $\widehat{S}_{ij}$ and $\widehat{\Omega}_{ij}$ are the normalized mean rates of strain and rotation:

$$\widehat{S}_{ij} \equiv \frac{1}{2}\frac{k}{\varepsilon}\left(\frac{\partial\langle U_i\rangle}{\partial x_j} + \frac{\partial\langle U_j\rangle}{\partial x_i}\right), \tag{4}$$

$$\widehat{\Omega}_{ij} \equiv \frac{1}{2}\frac{k}{\varepsilon}\left(\frac{\partial\langle U_i\rangle}{\partial x_j} - \frac{\partial\langle U_j\rangle}{\partial x_i}\right). \tag{5}$$
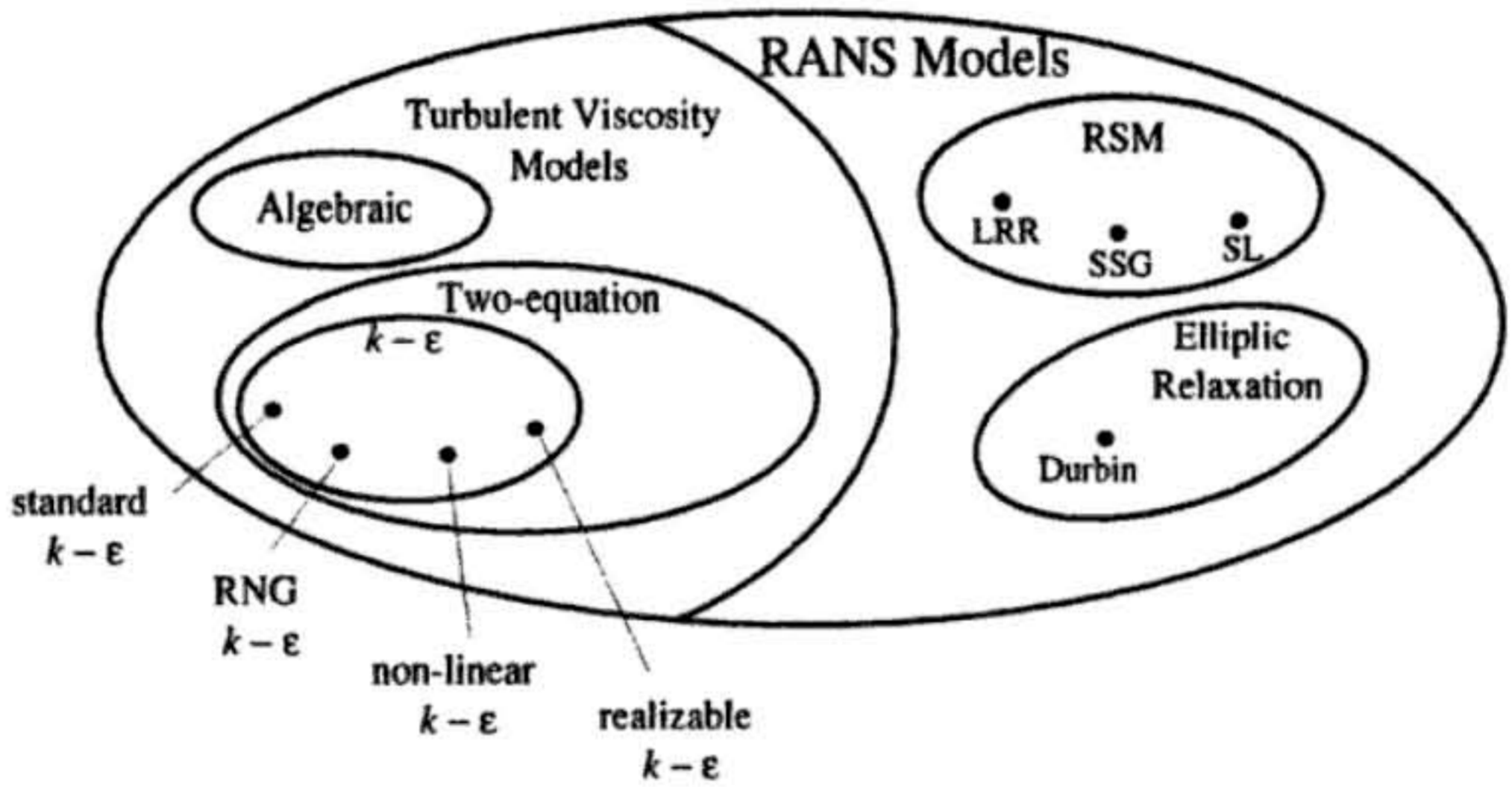

Figure 5. Sketches of particular models as members of subclasses and classes of models.

A particular model within the class corresponds to a particular specification of the constants C_μ, σ_k and σ_ε, of the non-dimensional scalar function $S_\varepsilon(\widehat{\mathbf{S}}, \widehat{\boldsymbol{\Omega}})$, and of the symmetric non-dimensional tensor function $\mathcal{F}_{ij}(\widehat{\mathbf{S}}, \widehat{\boldsymbol{\Omega}})$. For example, the specifications for the standard k-ε model are:

$$C_\mu = 0.09, \;\; \sigma_k = 1.0, \;\; \sigma_\varepsilon = 1.3, \tag{6}$$

$$\mathcal{F}_{ij} = \frac{2}{3}\delta_{ij} - 2C_\mu\widehat{S}_{ij}, \tag{7}$$

$$S_\varepsilon = 2C_\mu C_{\varepsilon 1}\widehat{S}_{ij}\widehat{S}_{ij} - C_{\varepsilon 2}, \tag{8}$$

with

$$C_{\varepsilon 1} = 1.44 \;\; \text{and} \;\; C_{\varepsilon 2} = 1.92. \tag{9}$$

It is important to appreciate that the assessment of *particular models* and of *classes of models* must be based on qualitatively different arguments.

For example, a comparison between experimental data and model calculations may reveal that (for the flow considered) the standard k-ε model calculation of the mean velocity is in error by up to 40%. Nothing can be deduced from this observation about the performance of *all* k-ε models.

In the absence of mean velocity gradients, the function $\mathcal{F}_{ij}$ is inevitably isotropic (e.g., $\mathcal{F}_{ij} = \frac{2}{3}\delta_{ij}$). Consequently *no k-ε* model (indeed, no turbulent viscosity model) can represent the Reynolds-stress anisotropy in decaying, anisotropic grid turbulence.

A quest for better models can proceed in two distinct directions. One is to seek higher levels of closure, so that the class of model considered can represent more turbulence phenomena. For example, a Reynolds-stress level may be chosen to overcome the turbulent viscosity models' inability to represent anisotropic decaying turbulence. Or a structure-based level (Reynolds *et al.*, 1995; Van Slooten *et al.*, 1997) may be chosen to overcome Reynolds-stress models' inability to calculate rapid distortions with mean rotation. Valuable though quests in this direction may be, it should be appreciated that:

a) any level of statistical closure provides only a partial representation of the turbulence, and therefore it cannot account for all turbulence behaviors

b) even though a level of closure is known to have limitations, a model at that level may nevertheless be able to calculate a range of flows to useful accuracy.

The second direction that can be taken in the quest for better models is to seek particular models within each class which are more accurate than current models. In the context of engineering turbulence model calculations (for most inert flows), essentially all of the progress that has been made in the last 25 years has been in this direction. The class of two-equation, k-ε, and Reynolds stress models had certainly been identified by 1972 (e.g., Jones *et al.* (1972); Hanjalic *et al.* (1972)). The newer k-ε models (e.g., RNG, non-linear, realizable) and Reynolds-stress models (e.g., LRR, Shih-Lumley, SSG) represent different particular models within the same class.

6. Towards Optimal Turbulence Models

It is argued above that the full range of models will be in use for the next 20 years—from two-equation to LES, with RANS models being dominant. It would be extremely valuable, therefore, to determine the *optimal model* within each class, and to have a general methodology for doing so. We offer the following four-step definition of an optimal model.

1. The *class of models* considered is defined by a set of equations (e.g., Eqs. 1–3) involving *free constants* (e.g., $C_\mu, \sigma_k, \sigma_\varepsilon$) and *free coefficients*

(e.g., $\mathcal{F}_{ij}$ and S_ε).

2. A *range of flows* is considered for which there are accurate experimental data.

3. For every model within the class, the *error* ϵ is defined as a specified weighted difference between the model calculation and the experimental data.

4. The *optimal model* corresponds to the specification of the free constants and coefficients[1] that minimizes the error ϵ.

It is all but certain that all current models are sub-optimal. Taking the standard k-ε model as an example, it has been known for 25 years (Rodi, 1972) that C_μ is more appropriately a non-trivial function of $\widehat{\mathbf{S}}$ and $\widehat{\mathbf{\Omega}}$ than a constant. While functional forms of C_μ have been proposed (e.g., Pope (1975)) there is no consensus on a near-optimal form. A methodology for determining optimal models would be able to determine the appropriate specifications, and would determine how far from optimal current models are.

Model coefficients are often specified by assuming them to be constant, and by determining the value of the constant by reference to a single observation. For example $C_\mu = 0.09$ stems from the observation (in turbulent shear flow) $C_\mu^{\frac{1}{2}} = |\langle u_1 u_2 \rangle|/k \approx 0.3$. Alternatively, a simple functional form is assumed, which is constructed or constrained to behave correctly in extreme limiting circumstances, such as rapid distortion or two-component or two-dimensional turbulence. Again, the numerical parameters in the assumed functional form are typically determined by reference to a few simple observations, often in homogeneous turbulence. Arguably, the relatively slow progress in Reynolds-stress modeling is due to overly restrictive functional forms for the coefficients, and to the lack of a methodology for determining near-optimal values of the free parameters.

There are strong theoretical arguments showing that some classes of models are superior to others: non-linear viscosity models are superior to linear viscosity models; Reynolds stress models are superior to turbulent viscosity models; models with more than one scale equation are superior to those with a single scale equation (e.g., the ε equation). This superiority is manifest in that the error ϵ incurred by the optimal model in the "superior" class, is less than that incurred by the optimal model in the "inferior" class. But of course, through an abysmal choice of the free coefficients, a model in the "superior" class can perform much worse than a reasonable model in the "inferior" class. Indeed, it has often been the case that in the early years of their development, particular models in a "superior" class are found

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[1]With coefficients—as opposed to constants—the minimization problem is under-determined, and additional smoothness conditions on the coefficients are appropriate.

not to be convincing improvements over the best models in the "inferior" class. In the assessment cited above—"stress-transport models ... appear somewhat better than eddy-viscosity models"—we believe that the qualifier is "somewhat" rather than "decisively" because current Reynolds-stress models are far from optimal.

The development of a methodology to determine optimal turbulence models (as defined above) is a challenge for future research. The constants in the generalized Langevin model (GLM) of Haworth et al. (1986a) were determined by the four-step optimization procedure described above; but no work on a general methodology for determining optimal turbulence-model coefficients is known to the author.

7. CONCLUSIONS

We have considered the future of turbulent flow computations in the context of engineering problems. The principal points of this paper—many of them conjectures—are as follows:

1. There is a very broad range of turbulent flow problems, and it is valuable (now and into the future) to have a broad range of turbulence modeling approaches.

2. Over the next 20 years there will be increased usage of turbulence models, dominantly RANS models.

3. There will be more usage of models that partially represent unsteady turbulent motions; more by a transition from RANS to unsteady RANS and beyond rather then by a jump to LES.

4. The projected increase in computer power (a factor of 30 in 10 years, 1000 in 20 years) will be used more on the first 6 items enumerated in Section 3.2 than on much more computationally-demanding approaches such as LES.

5. There would be great value in a methodology for determining optimal turbulence models at different levels of closure—as defined in Section 6.

6. The observed marginal superiority of Reynolds stress models over turbulent viscosity models—in contrast to their marked theoretical superiority—is attributed to existing Reynolds stress models being far from optimal.

7. Challenges for future research include

    (a) developing a general methodology for determining optimal turbulence models

    (b) developing unified models that can be applied (and are accurate) at any specified level of resolution of the unsteady turbulent motions (from RANS to LES to DNS)

(c) developing models for complex physical and chemical processes (e.g., combustion, multi-phase flow) for use in "partial resolution models."

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