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A more general effective-viscosity hypothesis

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A discussion of the applicability of an effective-viscosity approach to turbulent flow suggests that there are flow situations where the approach is valid and yet present hypotheses fail. The general form of an effective-viscosity formulation is shown to be a finite tensor polynomial. For two-dimensional flows, the coefficients of this polynomial are evaluated from the modelled Reynolds-stress equations of Launder, Reece & Rodi (1975). The advantage of the proposed effective-viscosity formulation, equation (4.3), over isotropic-viscosity hypotheses is that the whole velocity-gradient tensor affects the predicted Reynolds stresses. Two notable consequences of this are that (i) the complete Reynolds-stress tensor is realistically modelled and (ii) the influence of streamline curvature on the Reynolds stresses is incorporated.

1. Introduction

An effective-viscosity hypothesis relates the Reynolds stresses solely to the rates of strain of the fluid and to scalar quantities. The validity of such a hypothesis depends upon two conditions: first, the flow must be such that the Reynolds stresses are solely a function of those quantities considered and, second, the predicted Reynolds stresses must reflect experimental observations. Clearly, if the first condition is satisfied, then an effective-viscosity hypothesis exists that will satisfy the second. The object of this work is to obtain a more realistic effective-viscosity hypothesis than those currently employed.

The first effective-viscosity hypothesis was proposed by Boussinesq (1877):

$$-\overline{u_1 u_2} = \mu_{\text{eff}} U_{1,2} \tag{1.1}$$

(where $U_{1,2}$ is the only non-zero (mean) velocity gradient and μ_{eff} is the effective viscosity). This formula has been used with considerable success by, among others, Ng (1971) for wall boundary layers and by Rodi (1972) for free shear flows. However, Bradshaw (1973) has observed that the Boussinesq hypothesis fails for boundary layers over curved surfaces and inferred that this failure is due to the form of the stress–strain relation rather than to the inapplicability of an effective-viscosity approach.

For flows in which more than one Reynolds stress is needed to close the mean momentum equations, the Boussinesq hypothesis may be generalized to give the isotropic-viscosity assumption

$$\overline{u_i u_j} = \tfrac{2}{3}k\delta_{ij} - \mu_{\text{eff}}(U_{i,j} + U_{j,i}), \tag{1.2}$$

where k is the turbulence kinetic energy. While (1.2) has been used, with success, in recirculating flows (e.g. Runchal & Spalding 1971), it cannot be held to represent the Reynolds-stress tensor. For example, in nearly homogeneous shear flow (where, as will be shown in the next section, an effective-viscosity approach is valid) Champagne, Harris & Corrsin (1970) measured

$$a_{11} = 0{\cdot}3, \quad a_{22} = -0{\cdot}18, \quad a_{33} = -0{\cdot}12, \quad a_{12} = 0{\cdot}33,$$

where

$$a_{ij} = \overline{u_i u_j}/k - \tfrac{2}{3}\delta_{ij},$$

whereas, at best, (1.2) will predict

$$a_{11} = a_{22} = a_{33} = 0, \quad a_{12} = 0{\cdot}33.$$

Thus the mechanism that causes the inequality of the normal stresses cannot be accounted for with an isotropic-viscosity hypothesis. However, it was argued above that, in a situation where an effective-viscosity approach is valid, there is an effective-viscosity hypothesis that will predict the Reynolds stresses correctly.

The failure of isotropic viscosity hypotheses to give correct predictions of many flow situations is due either to inapplicability of an effective-viscosity approach or to inadequacy of the isotropic-viscosity hypotheses. By formulating an improved effective-viscosity hypothesis, it is the object of this work to remove the latter cause of failure.

The approach adopted in the next two sections, that of formulating a constitutive relation for the Reynolds stresses, is similar to that employed by Lumley (1970). First the quantities to be included in the constitutive relation are determined and then it is shown that the relation between these quantities may be expressed as a finite tensor polynomial to form the general effective-viscosity hypothesis. In forming the tensor polynomial Lumley (1970) made illicit use of the alternating tensor density and so the result and some of the conclusions based upon it were incorrect.

2. The effective-viscosity approach

The basic assumption of an effective-viscosity hypothesis is that the Reynolds stresses are uniquely related to the rates of strain and local scalar quantities. As this assumption is not valid for all flows, it is necessary to define the restricted class of flows for which it is valid. In addition, in order to effect closure, the number of scalar quantities that need be considered must be limited.

An effective-viscosity assumption implies that the stresses are determined locally whereas the exact Reynolds-stress equations show that they may be convected by both the mean and fluctuating velocities. For an effective-viscosity assumption to be valid, these transport terms must be negligible. It is readily seen, from the transport equations, that the triple correlation (which accounts for turbulent convection) will be zero only when the Reynolds stresses and consequently the rates of strain are homogeneous. Thus homogeneity of the rates of strain is a necessary condition for the effective-viscosity approach to be valid.

Lumley (1970) shows that the mean velocity field and boundary values of the fluctuating velocity are sufficient to determine the Reynolds stresses and assumes that, far from boundaries, the boundary conditions serve at most to set the levels of the time and length scales. With this assumption it follows that in a homogeneous flow (where the rates of strain contain all the information about the velocity field) the Reynolds stresses are a function of the rates of strain and scaling parameters only.

On dimensional grounds, at least two scaling parameters are needed to relate the Reynolds stresses to the rates of strain. These two scales may be chosen as a velocity scale v and a time scale τ; v^2 provides the dimensions of stress and τ may be used to non-dimensionalize the rate of strain. Two scaling parameters that may be derived from the mean velocity field are

$$v_m = (U_i U_i)^{\frac{1}{2}}, \quad \tau_m = (U_{i,j} U_{i,j})^{-\frac{1}{2}}. \tag{2.1), (2.2}$$

However v_m may not be chosen as the velocity scale because, although U_i changes under a Galilean transformation, $\overline{u_i u_j}$ does not. Also since the macro time scale of turbulence, defined by

$$\tau_t = k/\epsilon, \tag{2.3}$$

where ϵ is the turbulence dissipation rate, has been found to be independent of τ_m in simple shear flows (Lumley 1970), the two scaling parameters must be independent of the mean velocity field. Two scaling parameters are sufficient provided that all macroscales of turbulence are proportional; this assumption can be justified only at high Reynolds numbers, when any influence of the laminar viscosity may be excluded. Various authors' proposals for the two scaling parameters are given in Launder & Spalding (1972, p. 95). A convenient choice, and the one employed below, is that of k and ϵ. The fact that τ_t and τ_m are independent in any real flow situation means that the flow cannot be homogeneous and consequently some transport of Reynolds stresses occurs. However, in nearly homogeneous flows, where local effects dominate transport effects, an effective-viscosity hypothesis may provide an adequate representation of the Reynolds stresses.

The restrictions, assumptions and conclusions of the above argument may be stated thus: for a high Reynolds number nearly homogeneous flow, the Reynolds stresses are uniquely related to the rates of strain and two independent scaling parameters, provided that all macroscales are proportional and that the boundary conditions affect only the scaling parameters.

3. The general effective-viscosity hypothesis

In §2, physical arguments were employed to limit the number of quantities needed to determine the Reynolds stresses. In this section, by applying dimensional analysis, imposing invariance under co-ordinate transformation and exploiting the tensor properties of $U_{i,j}$ and $\overline{u_i u_j}$, the form of the general stress–strain relation will be deduced.

The two scaling parameters k and ϵ may be used to normalize the Reynolds stresses and the rates of strain as follows:

$$a_{ij} = \overline{u_i u_j}/k - \tfrac{2}{3}\delta_{ij}, \tag{3.1}$$

$$s_{ij} = \tfrac{1}{2}(k/\epsilon)(U_{i,j} + U_{j,i}), \tag{3.2}$$

$$\omega_{ij} = \tfrac{1}{2}(k/\epsilon)(U_{i,j} - U_{j,i}). \tag{3.3}$$

$\mathbf{a}$ and $\mathbf{s}$ are non-dimensional symmetric tensors with zero trace ($s_{ii} = 0$ owing to incompressibility) and $\boldsymbol{\omega}$ is non-dimensional and antisymmetric. The task of determining $\overline{u_i u_j}$ is equivalent to that of determining a_{ij} and, on dimensional grounds, s_{ij} and ω_{ij} contain all the information given by k, ϵ and $U_{i,j}$. Thus

$$a_{ij} = a_{ij}(\mathbf{s}, \boldsymbol{\omega}). \tag{3.4}$$

In continuum mechanics the assumption of material difference is frequently made in order to remove the dependence of $\mathbf{a}$ upon $\boldsymbol{\omega}$, however, as indicated by Lumley (1970), such an assumption is unfounded in the present context. For simplicity the following abbreviated notation will be introduced:

$$\mathbf{s}\boldsymbol{\omega} = s_{ik}\omega_{kj}, \quad \mathbf{s}\boldsymbol{\omega}\,\mathbf{s}\boldsymbol{\omega} = s_{ik}\omega_{kl}s_{lm}\omega_{mj}, \quad \text{etc.,}$$
$$\mathbf{s}^2 = s_{ik}s_{kj}, \quad \{\mathbf{s}^2\} = s_{ik}s_{ki}, \quad \text{etc.,} \quad \mathbf{I} = \delta_{ij}.$$

(This is equivalent to considering the matrices associated with the tensors.)

The most general expression for (3.4) is an infinite tensor polynomial:

$$\mathbf{a} = \prod_{i=1}^{\infty} \sum_{\alpha_i=0}^{\infty} \prod_{j=1}^{\infty} \sum_{\beta_j=0}^{\infty} G^{\alpha_1,\alpha_2\ldots}_{\beta_1,\beta_2\ldots}\, \mathbf{s}^{\alpha_1}\boldsymbol{\omega}^{\beta_1}\mathbf{s}^{\alpha_2}\boldsymbol{\omega}^{\beta_2}\ldots, \tag{3.5}$$

where the coefficients G may be functions of the invariants $\{\mathbf{s}^{\alpha_1}\boldsymbol{\omega}^{\beta_1}\mathbf{s}^{\alpha_2}\boldsymbol{\omega}^{\beta_2}\ldots\}$.

Fortunately, owing to the Cayley–Hamilton theorem, the number of independent invariants and linearly independent second-order tensors that may be formed from $\mathbf{s}$ and $\boldsymbol{\omega}$ is finite. This means that the infinite polynomial (3.5) may be expressed as a finite polynomial and that the coefficients G are functions of a finite number of invariants. Since $\mathbf{a}$ is symmetric and has zero trace, only independent tensors with these properties need be considered.

For flows in which the velocity and the variation of mean quantities in one co-ordinate direction are zero, $\mathbf{s}$ and $\boldsymbol{\omega}$ may be taken as two-dimensional tensors. As the number of linearly independent tensors and invariants in two dimensions is significantly less than in the general case, the two- and three-dimensional forms are treated separately.

The independent tensors and invariants are determined in appendix A. For two-dimensional flows there are only three linearly independent tensors $\mathbf{T}$ which are symmetric and have zero trace:

$$\mathbf{T}^0 = \tfrac{1}{3}\mathbf{I}_3 - \tfrac{1}{2}\mathbf{I}_2,\dagger \quad \mathbf{T}^1 = \mathbf{s}, \quad \mathbf{T}^2 = \mathbf{s}\boldsymbol{\omega} - \boldsymbol{\omega}\mathbf{s}.$$

There are two non-zero independent invariants:

$$\{\mathbf{s}^2\}, \quad \{\boldsymbol{\omega}^2\}.$$

In the general three-dimensional case there are ten tensors and five invariants:

$$\begin{aligned}
\mathbf{T}^1 &= \mathbf{s}, & \mathbf{T}^2 &= \mathbf{s}\boldsymbol{\omega} - \boldsymbol{\omega}\mathbf{s},\\
\mathbf{T}^3 &= \mathbf{s}^2 - \tfrac{1}{3}\mathbf{I}\{\mathbf{s}^2\}, & \mathbf{T}^4 &= \boldsymbol{\omega}^2 - \tfrac{1}{3}\mathbf{I}\{\boldsymbol{\omega}^2\},\\
\mathbf{T}^5 &= \boldsymbol{\omega}\mathbf{s}^2 - \mathbf{s}^2\boldsymbol{\omega}, & \mathbf{T}^6 &= \boldsymbol{\omega}^2\mathbf{s} + \mathbf{s}\boldsymbol{\omega}^2 - \tfrac{2}{3}\mathbf{I}\{\mathbf{s}\boldsymbol{\omega}^2\},\\
\mathbf{T}^7 &= \boldsymbol{\omega}\mathbf{s}\boldsymbol{\omega}^2 - \boldsymbol{\omega}^2\mathbf{s}\boldsymbol{\omega}, & \mathbf{T}^8 &= \mathbf{s}\boldsymbol{\omega}\mathbf{s}^2 - \mathbf{s}^2\boldsymbol{\omega}\mathbf{s},\\
\mathbf{T}^9 &= \boldsymbol{\omega}^2\mathbf{s}^2 + \mathbf{s}^2\boldsymbol{\omega}^2 - \tfrac{2}{3}\mathbf{I}\{\mathbf{s}^2\boldsymbol{\omega}^2\}, & \mathbf{T}^{10} &= \boldsymbol{\omega}\mathbf{s}^2\boldsymbol{\omega}^2 - \boldsymbol{\omega}^2\mathbf{s}^2\boldsymbol{\omega}
\end{aligned}$$

† $\mathbf{I}_2$ and $\mathbf{I}_3$ are the Kronecker deltas appropriate to two and three dimensions respectively. The propriety of the use of $\mathbf{I}_2$ is verified in appendix A.

and $\{\mathbf{s}^2\}, \quad \{\boldsymbol{\omega}^2\}, \quad \{\mathbf{s}^3\}, \quad \{\boldsymbol{\omega}^2\mathbf{s}\}, \quad \{\boldsymbol{\omega}^2\mathbf{s}^2\}.$

The infinite tensor polynomial (3.5) may now be expressed in the closed form

$$\mathbf{a} = \sum_{\lambda} G^{\lambda}\mathbf{T}^{\lambda} \tag{3.6}$$

($0 \leqslant \lambda \leqslant 2$ in two dimensions; $1 \leqslant \lambda \leqslant 10$ in three), or reverting to the unnormalized Reynolds stress, the general expression for any (tensor invariant) effective-viscosity hypothesis is

$$\overline{u_i u_j} = \tfrac{2}{3}k\delta_{ij} + k\sum_{\lambda} G^{\lambda}T^{\lambda}_{ij}. \tag{3.7}$$

The significance of (3.6) is that the Reynolds stresses are known functions of a finite number of known tensors and the same number of unknown scalars. The unknown scalars are in turn functions of a finite number of known invariants. For example, the task of formulating an effective-viscosity hypothesis for two-dimensional flows has been reduced to that of determining three scalars which may be functions of only two invariants.

4. Proposed effective-viscosity hypothesis

In order to complete the effective-viscosity hypothesis, the unknown functions G appearing in (3.6) must be determined. This will be done for two-dimensional flows by relating the general effective-viscosity hypothesis to the modelled Reynolds-stress equation of Launder *et al.* (1975).

In order to reduce this equation to one solely in terms of Reynolds stresses and velocity gradients it is necessary to model the transport resulting from small departures from homogeneity. This is done through the algebraic stress model suggested by Rodi (1972):

$$\text{transport of } \overline{u_i u_j} \approx (\overline{u_i u_j}/k)\,(\text{transport of } k) = (\overline{u_i u_j}/k)\,(P-\epsilon), \tag{4.1}$$

where P is the production rate of turbulence kinetic energy. Applying (4.1) to the Reynolds-stress equation of Launder *et al.* (1975) gives

$$\begin{aligned}(\overline{u_i u_j}/k)\,(P-\epsilon) = {} & -\overline{u_i u_k}\,U_{j,k} - \overline{u_j u_k}\,U_{i,k} - \tfrac{2}{3}\epsilon\delta_{ij} - C_1(\epsilon/k)\,(\overline{u_i u_j} - \tfrac{2}{3}k\delta_{ij}) \\ & + \tfrac{1}{11}(C_2+8)\,(\overline{u_i u_k}\,U_{j,k} + \overline{u_j u_k}\,U_{i,k} + \tfrac{2}{3}P\delta_{ij}) \\ & - \tfrac{1}{55}(30C_2-2)\,k(U_{i,j}+U_{j,i}) \\ & + \tfrac{1}{11}(8C_2-2)\,(\overline{u_i u_k}\,U_{k,j} + \overline{u_j u_k}\,U_{k,i} + \tfrac{2}{3}P\delta_{ij}),\end{aligned}$$

or, in the present notation,

$$\mathbf{a} = -g[b_1\mathbf{s} + b_2(\mathbf{as}+\mathbf{sa} - 2\mathbf{I}_3\{\mathbf{as}\}) - b_3(\mathbf{a}\boldsymbol{\omega} - \boldsymbol{\omega}\mathbf{a})], \tag{4.2}$$

where $\quad b_1 = \tfrac{8}{15}, \quad b_2 = \tfrac{1}{11}(5-9C_2), \quad b_3 = \tfrac{1}{11}(7C_2+1), \quad g = (C_1 + P/\epsilon - 1)^{-1}.$

C_1 and C_2 are constants appearing in the modelled Reynolds-stress equation, for which values of 1·5 and 0·4, respectively, were suggested.

Equation (4.2), which is a set of simultaneous equations, may be expressed as an explicit relation of the form of (3.6) (see appendix B):

$$\mathbf{a} = -2C_\mu[\mathbf{s} + gb_3(\mathbf{s}\boldsymbol{\omega} - \boldsymbol{\omega}\mathbf{s}) + gb_2\{\mathbf{s}^2\}(\tfrac{2}{3}\mathbf{I}_3 - \mathbf{I}_2)], \tag{4.3}$$

where
$$C_\mu = \tfrac{1}{2}b_1 g(1 - 2\{\boldsymbol{\omega}^2\}b_3^2 g^2 - \tfrac{2}{3}b_2^2 g^2\{\mathbf{s}^2\})^{-1}. \tag{4.4}$$

The proposed effective-viscosity hypothesis (4.3) is of the form of (3.6) with

$$G^0 = 4C_\mu gb_2\{\mathbf{s}^2\}, \quad G^1 = -2C_\mu, \quad G^2 = -2C_\mu gb_3.$$

The Reynolds-stress equation of Launder *et al.* (1975), the algebraic stress model and the proposed effective-viscosity hypothesis will all predict the same Reynolds stresses provided that the transport is well modelled by (4.1).

For practical purposes (4.3) may be simplified. If the Reynolds stresses are only required to effect closure in the mean momentum equations and to calculate the production rate of kinetic energy, then only the part $(\overline{u_i u_j})_A$ of $\overline{u_i u_j}$ that is anisotropic in the plane normal to the direction of homogeneity need be specified. Reverting to standard notation, (4.3) may be written as

$$(\overline{u_i u_j})_A = -C_\mu \frac{k^2}{\epsilon}\left[(U_{i,j}+U_{j,i})+gb_3\frac{k}{\epsilon}(U_{l,i}U_{l,j}-U_{i,l}U_{j,l})\right]. \qquad (4.5)$$

A solution of the mean momentum equations using $\overline{u_i u_j}$ as given by (4.3) or $(\overline{u_i u_j})_A$ as given by (4.5) will produce identical results. Equation (4.5) differs from an isotropic-viscosity hypothesis only by the inclusion of the last term.

5. Discussion

The advantages of the proposed effective-viscosity hypothesis over isotropic-viscosity hypotheses may be demonstrated by comparing the values of the Reynolds stress predicted by each in a simple flow. For this purpose a flow in which U_1 is the only non-zero component of the mean velocity and x_2 the only direction of variation is considered below.

For this flow the general effective-viscosity formulation (3.6) gives

$$\begin{aligned} a_{11} &= -\tfrac{1}{6}G^0 && -\tfrac{1}{2}G^2[(k/\epsilon)\,U_{1,2}]^2,\\ a_{22} &= -\tfrac{1}{6}G^0 && +\tfrac{1}{2}G^2[(k/\epsilon)\,U_{1,2}]^2,\\ a_{33} &= \tfrac{1}{3}G^0, &&\\ a_{12} &= \tfrac{1}{2}G^1(k/\epsilon)\,U_{1,2}, &&\\ a_{13} &= a_{23} = 0. && \end{aligned}$$

It may be seen that G^1 does not influence the normal stresses and that G^0 and G^2 do not influence the shear stress. A finite value of G^2 causes a_{11} and a_{22} to differ and G^0 enables a_{33} to depart from zero. Clearly, if G^0 and G^2 are set to zero, as is the case in an isotropic-viscosity hypothesis, then the observed differences between the normal stresses cannot be predicted. This inherent deficiency in isotropic-viscosity hypotheses suggests that they will provide an inadequate closure for more complex flows, where more than one component of the Reynolds stress is required to close the mean momentum equations.

The choice of G^1, and consequently C_μ, is of paramount importance as it dictates the predicted shear-stress level. Figure 1 shows the variation of C_μ, given by (4.4), as a function of σ and Ω, where $\sigma \equiv (\tfrac{1}{2}\{\mathbf{s}^2\})^{\frac{1}{2}}$ and $\Omega \equiv (-\tfrac{1}{2}\{\boldsymbol{\omega}^2\})^{\frac{1}{2}}$.

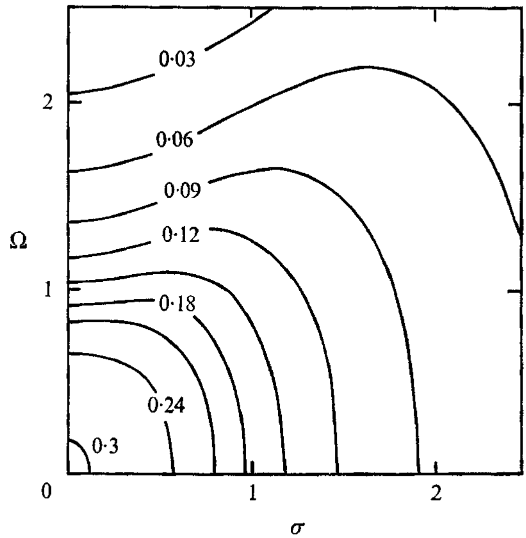

FIGURE 1. Variation of C_μ with σ and Ω.

This variation of C_μ may be compared with previous suggestions:

Prandtl–Kolmogorov	C_μ = constant,
Bradshaw, Ferriss & Atwell (1967)†	$C_\mu \propto \sigma^{-1}$,
Rodi (1972)	$C_\mu = C_\mu(P/\epsilon)$.

While the two latter expressions are in accord with the present proposal in that they predict a decrease in C_μ with increasing σ, none of the above expressions allows for any dependence of C_μ upon the rotation invariant Ω. This omission is tantamount to assuming that the Reynolds stresses are materially indifferent; that is, to assuming that the Reynolds stresses are unaffected by solid-body rotations. The use of this unfounded assumption is most likely responsible for the short-comings of these isotropic-viscosity hypotheses in predicting flows with streamline curvature.

While the proposed effective-viscosity hypothesis has advantages over isotropic hypotheses, its predictions are identical to those of the algebraic stress model and it has the disadvantage of being restricted to two-dimensional flows. (The three-dimensional form is so intractable as to be of no value.) However the solution of the mean momentum equations expressed in terms of the effective-viscosity formula offers two advantages over the use of the algebraic stress relation. First, the inter-relation between strain and stress is retained within the differential equation, thus increasing numerical stability, and second, the time-consuming solution of the algebraic stress (simultaneous) equations is not needed.

A solution procedure based on an isotropic-viscosity hypothesis may use the proposed formulation simply by evaluating C_μ through (4.4) and including the additional term in (4.5).

† $\mathbf{a} \propto \mathbf{s}/\sigma$ may be considered as a tensor-invariant expression of Bradshaw's hypothesis a_{12} = constant.

Appendix A

Tensor functions in two dimensions

Consider three arbitrary two-dimensional second-order tensors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. The Cayley–Hamilton theorem states that

$$\mathbf{c}^2 = \mathbf{c}\{\mathbf{c}\} - \tfrac{1}{2}\mathbf{I}_2(\{\mathbf{c}\}^2 - \{\mathbf{c}^2\}). \tag{A 1}$$

Substituting $\mathbf{c} = \mathbf{a} + \mathbf{b}$ into (A 1) gives

$$\mathbf{ab} + \mathbf{ba} = \mathbf{a}\{\mathbf{b}\} + \mathbf{b}\{\mathbf{a}\} - \mathbf{I}_2(\{\mathbf{a}\}\{\mathbf{b}\} - \{\mathbf{ab}\}) \tag{A 2}$$

and post multiplying (A 2) by $\mathbf{b}$

$$\mathbf{bab} = \tfrac{1}{2}(\mathbf{a} - \mathbf{I}_2\{\mathbf{a}\})(\{\mathbf{b}\}^2 - \{\mathbf{b}^2\}) + \mathbf{b}\{\mathbf{ab}\}. \tag{A 3}$$

Provided that the transposes of $\mathbf{a}$ and $\mathbf{b}$ may be expressed as linear functions of $\mathbf{a}$ and $\mathbf{b}$, a general polynomial term in $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\mathbf{P} = \mathbf{a}^{\alpha_1}\mathbf{b}^{\beta_1}\mathbf{a}^{\alpha_2}\mathbf{b}^{\beta_2}\ldots\mathbf{a}^{\alpha_n}\mathbf{b}^{\beta_m}, \tag{A 4}$$

where α_i and β_i may take any positive integer value. Define the extension of $\mathbf{P}$ as $n+m$ and the partial orders of $\mathbf{P}$ in $\mathbf{a}$ and $\mathbf{b}$ to be $\max\alpha_i$ and $\max\beta_i$ respectively.

If the partial order of $\mathbf{P}$ is greater than one, it may be reduced to a sum of polynomials of partial order one by repeated substitution of (A 1). If the extension of these polynomials is greater than two they may be reduced to a sum of polynomials of extension less than two by repeated application of (A 3). Of the two possible tensor functions of extension two ($\mathbf{ab}$ and $\mathbf{ba}$) only one is independent owing to (A 2). Thus the independent tensor functions are $\mathbf{I}_2$, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{ab}$ (or $\mathbf{ba}$).

Independent invariants of two second-order tensors in two dimensions

Any invariant may be formed by taking the trace of the general polynomial expression $\mathbf{P}$. If $\mathbf{P}$ may be reduced to polynomials of lesser partial order and extension, then $\{\mathbf{P}\}$ will be given by the trace of the reduced polynomials. Thus the independent invariants may be taken as the traces of all the linearly independent tensors and the invariants required to reduce the general polynomial, i.e. $\{\mathbf{a}\}$, $\{\mathbf{b}\}$, $\{\mathbf{ab}\}$, $\{\mathbf{a}^2\}$ and $\{\mathbf{b}^2\}$.

Application to the present situation

Section 3 requires the determination of all the independent tensor functions that are symmetric and have zero trace and the independent invariants that may be formed from $\mathbf{s}$ and $\boldsymbol{\omega}$. $\mathbf{s}$ is symmetric and has zero trace, $\boldsymbol{\omega}$ is antisymmetric.

In general the tensor functions will be $\mathbf{I}_2$, $\mathbf{s}$, $\boldsymbol{\omega}$ and $\mathbf{s}\boldsymbol{\omega}$. From these, only two tensor functions may be formed that are symmetric and have zero trace, viz. $\mathbf{s}$ and $\mathbf{s}\boldsymbol{\omega} - \boldsymbol{\omega}\mathbf{s}$. However, although the velocity field is two-dimensional, the Reynolds-stress tensor is not. The tensor functions must therefore be expressed as three-dimensional tensors, and the three-dimensional Kronecker delta $\mathbf{I}_3$ must be included. This leads to the further tensor function $\tfrac{1}{3}\mathbf{I}_3 - \tfrac{1}{2}\mathbf{I}_2$. Thus the complete set is $\tfrac{1}{3}\mathbf{I}_3 - \tfrac{1}{2}\mathbf{I}_2$, $\mathbf{s}$ and $\mathbf{s}\boldsymbol{\omega} - \boldsymbol{\omega}\mathbf{s}$.

Of the five possible independent invariants, only two are non-zero: $\{\mathbf{s}^2\}$ and

$\{\boldsymbol{\omega}^2\}$. Possible doubts concerning the practice of mixing two- and three-dimensional tensors can be dispelled. If, for example, the direction x_3 is the one in which there is no variation, then

$$\mathbf{I}_3 = \delta_{ij}^{(3)} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

$$\mathbf{I}_2 = \delta_{ij}^{(2)} = \begin{cases} 1, & i = j \neq 3, \\ 0, & i \neq j \quad \text{or} \quad i = j = 3. \end{cases}$$

Clearly $\mathbf{I}_2$ expressed in three dimensions is not an isotropic tensor in that it may change under a Cartesian transformation. However, as a preferential coordinate system has been chosen, it is sufficient that $\mathbf{I}_2$ transforms correctly within the restrictions of that system. That is, so long as $\mathbf{I}_2$ transforms as an isotropic tensor for all transformations in the x_1, x_2 plane, as it does, it is a properly defined and behaved quantity.

Independent tensors and invariants in three dimensions

The procedure used in two dimensions is also applicable to three dimensions. The procedure is reported by Spencer & Rivlin (1959, 1960) and the results for this situation are quoted in the text.

Appendix B. Evaluations of the functions G

Since the tensors $\mathbf{T}$ are all linearly independent functions of $\mathbf{s}$ and $\boldsymbol{\omega}$ it is possible to define scalar functions H and J as follows:

$$\mathbf{T}^\gamma \mathbf{s} + \mathbf{s}\mathbf{T}^\gamma - \tfrac{2}{3}\mathbf{I}_3\{\mathbf{s}\mathbf{T}^\gamma\} = \sum_\lambda {}^\gamma H^\lambda \mathbf{T}^\lambda, \tag{B 1}$$

$$\mathbf{T}^\gamma \boldsymbol{\omega} - \boldsymbol{\omega}\mathbf{T}^\gamma = \sum_\lambda {}^\gamma J^\lambda \mathbf{T}^\lambda. \tag{B 2}$$

Substituting (3.6), (B 1) and (B 2) into (4.2) gives

$$\sum_\gamma G^\gamma \mathbf{T}^\gamma = -g(b_1 \mathbf{T}^1 + b_2 \sum_\gamma G^\gamma \sum_\lambda {}^\gamma H^\lambda \mathbf{T}^\lambda - b_3 \sum_\gamma G^\gamma \sum_\lambda {}^\gamma J^\lambda \mathbf{T}^\lambda). \tag{B 3}$$

As the $\mathbf{T}$'s are independent, their coefficients on either side of (B 3) may be equated:

$$G^\gamma = -g(b_1 \delta_{1\gamma} + b_2 \sum_\lambda G^{\lambda\,\lambda} H^\gamma - b_3 \sum_\lambda G^{\lambda\,\lambda} J^\gamma). \tag{B 4}$$

Hence, having evaluated H and J with the aid of (A 1)–(A 3), there results a set of simultaneous equations for G:

$$G^0 = 2b_2 g\{\mathbf{s}^2\} G^1,$$
$$G^1 = -g(b_1 - \tfrac{1}{3}b_2 G^0 - 2b_3\{\boldsymbol{\omega}^2\} G^2),$$
$$G^2 = b_3 g G^1.$$

Solving for G and substituting in (3.6) gives

$$\mathbf{a} = -2C_\mu[\mathbf{s} + gb_3(\mathbf{s}\boldsymbol{\omega} - \boldsymbol{\omega}\mathbf{s}) + gb_2\{\mathbf{s}^2\}(\tfrac{2}{3}\mathbf{I}_3 - \mathbf{I}_2)],$$

where $$C_\mu = \tfrac{1}{2}gb_1(1 - 2g^2b_3^2\{\boldsymbol{\omega}^2\} - \tfrac{2}{3}b_2^2 g^2\{\mathbf{s}^2\})^{-1}.$$

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