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THE EVOLUTION OF SURFACES IN TURBULENCE

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Abstract—There are several phenomena that can be described to advantage in terms of surfaces within a turbulent fluid. Examples are: turbulent mixing (particularly at high Schmidt number); turbulent premixed flames; and, turbulent diffusion flames. These phenomena can (under appropriate conditions) be analyzed in terms of material surfaces, propagating surfaces, and constant-property surfaces, respectively. Deterministic and probabilistic equations are developed for the evolution of the local properties of these surfaces.

The local geometry of regular surfaces is described by the surface element properties: position; normal to the surface; principal curvatures and directions; and, fractional area increase. Exact evolution equations for these properties are derived which reveal the effects of various processes—straining, and surface propagation, for example. For material surfaces and simple propagating surfaces these equations are closed with respect to surface properties: that is, given the velocity field, the equations can be solved from specified initial conditions. The circumstances that can lead to a breakdown of regularity of an initially regular surface are determined.

The fundamental one-point Eulerian probabilistic descriptor of a regular surface is the surface density function. From this can be determined the expected surface-to-volume ratio and the joint probability density function of the surface properties. An exact evolution equation for the surface density function is derived and discussed. For material surfaces and simple propagating surfaces the only unknowns in this equation are statistics of the velocity field. These statistics can be modelled (via Langevin equations, for example) and then the surface-density-function equation can be solved by a Monte Carlo method.

1. INTRODUCTION

In most theoretical approaches to turbulence and turbulent combustion, the fundamental quantities considered are fluid properties (e.g. velocity, temperature, composition) at one or more points in space and time. But there are several mixing and reaction phenomena that can be described to advantage in terms of surfaces (or infinitesimal surface elements) within the fluid. We cite three examples to illustrate the three types of surfaces considered—*material surfaces, propagating surfaces,* and *constant-property surfaces.*

Consider the mixing of two bodies of fluid in turbulent motion that initially contain uniform but different concentrations of a contaminant. We consider the material surface that is initially coincident with the interface between the two bodies of fluid. Subsequently this material surface is convected, bent and strained by the turbulent motion. Because of the initial concentration difference across the material surface, a diffusive layer develops.

At early times the mixing near each point on the material surface can be well approximated as a transient one-dimensional diffusion process (normal to the surface) in a uniform strain field (see Fig. 1). With y being the coordinate normal to the surface, and $\phi(y, t)$ being the normalized concentration, the diffusion equation is

$$\frac{\partial \phi}{\partial t} + s_N y \frac{\partial \phi}{\partial y} = D \frac{\partial^2 \phi}{\partial y^2}, \tag{1.1}$$

where D is the diffusion coefficient and $s_N(t)$ is the rate of strain normal to the surface. The solution to this equation (with the boundary conditions indicated on the figure) is

$$\phi(y, t) = \int_{-\infty}^{y} \frac{1}{\sigma\sqrt{2\pi}} \exp(-\tfrac{1}{2}z^2/\sigma^2)\, dz, \tag{1.2}$$

where $\sigma(t)$ is given by

$$\sigma^2 = 2D \int_0^t \exp\left\{\int_{t'}^{t} 2s_N(t'')\, dt''\right\} dt'. \tag{1.3}$$

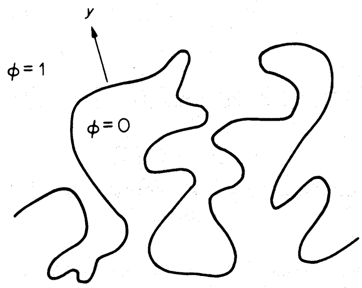

Fig. 1. Diffusive layer separating bodies of fluid with $\phi = 0$ and $\phi = 1$.

(This analysis is valid as long as σ is small compared to the Kolmogorov scale, to the radii of curvature of the surface, and to the distance to the closest intersection of the y axis with the material surface. It is most useful at high Schmidt number since then the growth of the diffusive layer is relatively slow.)

Thus the early stages of turbulent mixing can be completely described by the position, orientation, and strain history of each point on the material surface.

A similar approach to mixing problems has been taken by Batchelor [1] and Ottino *et al.* [2], and extended to reactive systems by Chella and Ottino [3] and Lundgren [4].

A *propagating surface* is defined to be a surface that propagates normal to itself at a speed w relative to the fluid. The speed w may vary over the surface but is positive everywhere. Thus a surface point with position $\mathbf{X}(t)$ moves with the velocity

$$\dot{\mathbf{X}}(t) = \mathbf{U}(\mathbf{X}[t], t) + w(\mathbf{X}[t])\mathbf{N}(\mathbf{X}[t]), \tag{1.4}$$

where $\mathbf{U}(\mathbf{x}, t)$ is the Eulerian velocity field, and $\mathbf{N}$ is the normal to the surface. (The sign of $\mathbf{N}$ determines the direction of propagation.)

The best example of a phenomenon that can be naturally described by a propagating surface is a turbulent premixed flame [5]. Provided the laminar flame thickness is much smaller than other length scales (e.g. the radii of curvature of the flame), then the flame can be regarded as a surface separating burnt and unburnt fluid. This flame surface is convected, bent and strained by the turbulence, and propagates normal to itself (and relative to the unburnt reactants just ahead) at the local flame speed, S_l (i.e., $w = S_l$). To a first approximation S_l is equal to the flame speed S_u of an unperturbed, plane laminar flame. To a second approximation [6] S_l depends on the flame stretch $\dot{S}$ and on the thermochemical properties of the mixture.

Flame stretch, introduced by Karlovitz *et al.* [7], is the local fractional rate of increase of surface area. Since the overall combustion rate is the product of the flame speed S_l and the flame area, the rate of change of flame area and hence the flame stretch $\dot{S}$ are important quantities whether or not S_l depends on $\dot{S}$. The flame stretch depends both on the rate of strain in the reactants and on the curvature of the surface [8, 9].

Thus in this example, as well as the position and orientation of the surface, its curvature is also an important property.

The third type of surface considered is a *constant-property surface.* Let $\phi(\mathbf{x}, t)$ be a property (e.g. temperature or concentration) that has a known evolution equation of the form

$$\frac{D\phi}{Dt} = \left(\frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x_i}\right)\phi = \Theta, \tag{1.5}$$

where the source Θ may contain functions and derivatives of ϕ. (It is assumed that Θ and the initial and boundary conditions are such that ϕ is twice differentiable.) Then with ϕ_0 being a constant, at time t the points $\mathbf{X}$ that satisfy the equation

$$\phi(\mathbf{X}, t) = \phi_0, \tag{1.6}$$

form a *constant-property surface.*

Again we draw an example from turbulent combustion, in this case a turbulent diffusion flame. According to simple theory [10, 11], reaction takes place only when the mixture fraction $\phi(\mathbf{x}, t)$ is very close to the stoichiometric value ϕ_0. That is, there is a reaction sheet surrounding the constant property surface defined by eqn (1.6). Recently, more elaborate theories have been proposed [12, 13] in which the properties of the reaction sheet depend upon the local straining. Marble and Broadwell [14] have developed a complete model of turbulent diffusion flames based on the notion of reacting surfaces.

The normal to a constant property surface is

$$\mathbf{N} = \nabla\phi / |\nabla\phi|, \tag{1.7}$$

and the surface propagates (relative to the fluid) at the speed

$$w = -\frac{\mathrm{D}\phi}{\mathrm{D}t} \Big/ |\nabla\phi|, \tag{1.8}$$

(see Gibson [15] and eqn (2.24)). In view of eqn (1.8), *all three types of surface can be regarded as constant-property surfaces or as propagating surfaces* (though the propagation speed w may not be positive). A material surface is a propagating surface with $w = 0$, or a constant-property surface with $\mathrm{D}\phi/\mathrm{D}t = 0$. A propagating surface can be treated as a constant-property surface with $\mathrm{D}\phi/\mathrm{D}t = -w|\nabla\phi|$; and a constant-property surface can be treated as a propagating surface with $w = -\mathrm{D}\phi/\mathrm{D}t/|\nabla\phi|$.

While a constant-property surface evolves in time, at any instant it is completely determined by the current property field $\phi(\mathbf{x}, t)$, independent of the surface's past history. Thus, rather than considering evolution equations for a constant-property surface, an alternative approach is to deduce the surface properties from the property field $\phi(\mathbf{x}, t)$ and its evolution. This approach, which has been extensively explored by Gibson [15], is most likely more useful, since a knowledge of the property field $\phi(\mathbf{x}, t)$ is needed in order to determine the propagation speed w [eqn (1.8)]. Consequently, here we give less consideration to constant-property surfaces than to material and propagating surfaces.

In this paper we begin to develop the exact deterministic and probabilistic equations governing the evolution of surfaces in turbulent flow. Our attention is confined to local properties of regular surfaces. The local properties of a surface element are: its position; the normal to the surface; the principal curvatures and directions; and, the fractional area increase of the element. These properties are described in the next section.

A regular surface [16] has finite curvature everywhere and has no self-intersections, critical points, or edges (at least within the fluid). We consider surfaces that are initially regular, and whether they remain regular is an important question that is addressed in Section 4. A material surface remains regular, but a propagating surface can develop singularities (infinite curvature) and self-intersections. A constant-property surface [eqn (1.6)] has critical points wherever the gradient of ϕ is zero on the surface, but is otherwise regular.

A surface can be defined implicitly through an equation of the form $\mathscr{F}(\mathbf{X}, t) = 0$ [cf. eqn (1.6)], or it can be defined explicitly as $\mathbf{X}(u, v, t)$, where u and v are parameters. Then the evolution of the whole surface is determined by $\partial\mathscr{F}/\partial t$ or $\partial\mathbf{X}/\partial t$. But in order to obtain a tractable probabilistic description, rather than considering the whole surface, we want to study the local surface properties listed above. A probabilistic description of the whole surface would resemble the functional formalism of Hopf [17] (see also Monin and Yaglom [18]) which has not proved tractable when applied to flows of interest. Instead, the one-point probabilistic description of local surface properties developed here is closely analogous to the pdf method [19] which has been successfully employed to calculate the properties of several turbulent flames (see, for example [20–22]).

Previous work on material surfaces has been limited to considering the stretching rate of a surface element and the related topic of material-line stretching [2, 23–25]. Brakke [26] and Sethian [27] considered the evolution of curvature in propagating surfaces, but in the absence of turbulence. In three respects the present work extends our knowledge of the evolution of surfaces in turbulence: propagating and constant-property surfaces (as well as material surfaces) are treated; the evolution of all local surface properties (including curvature) are considered; and, probabilistic as well as deterministic evolution equations are derived.

The local properties of a surface element are defined and described in the next section, and their evolution equations are presented and discussed in Section 3. The breakdown of regularity of the surfaces is considered in Section 4. The probabilistic description of surfaces in turbulent flow is introduced in Section 5. This is based on the *surface density function* F from which can be deduced the *expected surface area per unit volume* Σ, and the joint probability density function of the surface properties. Evolution equations for F and Σ are derived and discussed in Section 5.

This work has three principal uses:

(i) The concepts and exact equations developed here can help to guide experiments (e.g. [28]) and phenomenological models (e.g. [14]) involving surfaces.
(ii) In direct numerical simulations of turbulence, the deterministic surface evolution equations provide an alternative means of computing the evolution of surfaces.
(iii) The probabilistic description provides the theoretical basis for the stochastic modelling of the evolution of surfaces. The structure of the equation for F is the same as that of joint pdf equations [19] and similar stochastic modelling and Monte Carlo methods can be used.

2. SURFACE ELEMENT PROPERTIES

Figure 2 is a sketch of part of the surface at a fixed time t. The coordinates of the points on the surface are $\mathbf{X}(u, v, t)$, where u and v are local coordinates (not necessarily orthogonal) used to parametrize the surface. The unit normal vector to the surface is $\mathbf{N}(u, v, t)$. By assumption the surface is orientable. For a material surface the choice of sign of $\mathbf{N}(u, v, t)$ is immaterial; for a propagating surface $\mathbf{N}(u, v, t)$ is chosen to be in the direction of propagation; and, for a constant-property surface it is chosen to be in the direction of $\nabla\phi$.

We are interested in the surface in the neighborhood of an arbitrarily chosen surface point. A *surface point* is defined, first, by its location on the surface at a reference initial time t_0; and, second, by the specification that it remains on the surface by moving relative to the fluid (if at all) in the direction of the local normal to the surface. Thus the position $\mathbf{X}^{\circ}(t)$ of the surface point (denoted by O) originating from

$$\mathbf{X}^{\circ}(t_0) = \mathbf{X}(u_{\circ}, v_{\circ}, t_0), \tag{2.1}$$

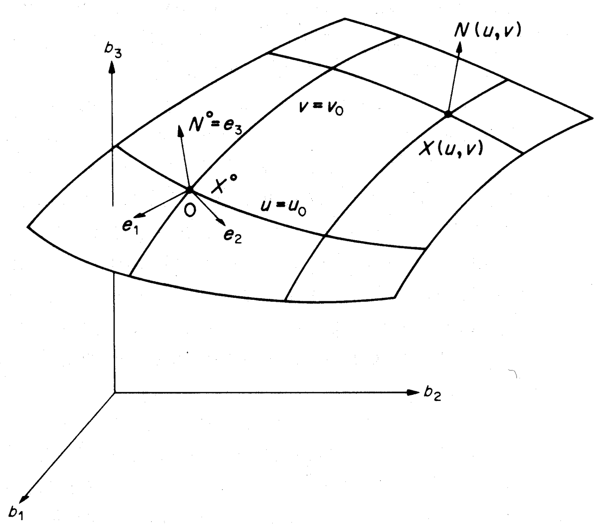

Fig. 2. Sketch of part of the surface showing the B and E coordinate systems.

evolves according to the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}^{\circ}(t) = \mathbf{U}(\mathbf{X}^{\circ}[t], t) + w(\mathbf{X}^{\circ}[t])\mathbf{N}^{\circ}(t), \tag{2.2}$$

where $\mathbf{N}^{\circ}$ is the normal at $\mathbf{X}^{\circ}$. For a material surface w is zero, and the surface point is also a material point or fluid particle. For a propagating surface w is the specified rate of propagation that may vary over the surface; and for a constant-property surface w is given by eqn (1.8).

This definition of a surface point needs to be refined when the fluid velocity is discontinuous across the surface. (In a premixed flame, the component of velocity normal to the surface is discontinuous across the surface but, to a first approximation, the tangential components are continuous [5, 6, 29].) Let $\mathbf{U}$ and $\mathbf{U}^-$ denote the fluid velocities on the positive and negative sides of the surface respectively (that is, the sides approached from $\mathbf{X} + |\varepsilon|\,\mathbf{N}$ and $\mathbf{X} - |\varepsilon|\,\mathbf{N}$, respectively). Then, by definition the surface point $\mathbf{X}^{\circ}(t)$ moves at a velocity $w(\mathbf{X}^{\circ}[t])\mathbf{N}^{\circ}(t)$ relative to the fluid on the *positive* side of the surface. Thus, eqn (2.2) remains the defining equation. (In the context of premixed flames, this definition is consistent with the convention of referring quantities to the side of the reactants.) Henceforth the velocity $\mathbf{U}$ and its derivatives $\partial U_i/\partial x_j$ and $\partial^2 U_i/\partial x_j\,\partial x_k$ refer to the fluid on the positive side of the surface, and are defined by a limiting process if need be.

The parametrization of the surface can be chosen at will, and does not affect the intrinsic surface properties. Having chosen the initial parametrization, it is convenient to let u and v remain constant following a surface point. Then, for the surface point originating from $\mathbf{X}(u, v, t_0)$, eqn (2.2) can be rewritten

$$\frac{\partial \mathbf{X}(u, v, t)}{\partial t} = \mathbf{U}(\mathbf{X}[u, v, t], t) + w(u, v, t)\mathbf{N}(u, v, t). \tag{2.3}$$

We shall study the evolution of the surface in the neighborhood of the surface point O, with location $\mathbf{X}^{\circ}(t)$ and normal $\mathbf{N}^{\circ}(t)$. Two Cartesian coordinate systems are used (see Fig. 2). The B coordinate system has fixed orthonormal basis vectors $\mathbf{b}_1, \mathbf{b}_2$ and $\mathbf{b}_3$. In this system, the coordinates of the surface point O are $X_i^{\circ}(t)$:

$$\mathbf{X}^{\circ}(t) = \mathbf{b}_i X_i^{\circ}(t). \tag{2.4}$$

The E coordinate system has its origin at O, and has time-dependent orthonormal basis vectors $\mathbf{e}_1(t)$, $\mathbf{e}_2(t)$ and $\mathbf{e}_3(t)$. The basis vector $\mathbf{e}_3(t)$ is the normal to the surface $\mathbf{N}^{\circ}(t)$, and hence $\mathbf{e}_1(t)$ and $\mathbf{e}_2(t)$ are in the tangent plane at O. The initial orientation of $\mathbf{e}_1(t_0)$ is arbitrary, but subsequently $\mathbf{e}_1(t)$ and $\mathbf{e}_2(t)$ are made to rotate with the angular velocity of the fluid in the tangent plane.

In the E coordinate system, a point on the surface has coordinates $y_i(u, v, t)$ relative to $\mathbf{X}^{\circ}(t)$:

$$\mathbf{X}(u, v, t) = \mathbf{X}^{\circ}(t) + \mathbf{e}_i y_i(u, v, t). \tag{2.5}$$

In a sufficiently small neighborhood of O, y_3 is a single-valued function of y_1 and y_2 (since, by assumption, the surface is regular). We can therefore introduce the *height function*

$$h(y_1, y_2, t) = y_3(u, v, t), \tag{2.6}$$

which measures the height of the surface above the tangent plane.

Since the surface is tangential to the $\mathbf{e}_1 - \mathbf{e}_2$ plane at $\mathbf{y} = 0$, we have

$$h(0, 0, t) = \left(\frac{\partial h}{\partial y_1}\right)_{\mathbf{y}=\mathrm{o}} = \left(\frac{\partial h}{\partial y_2}\right)_{\mathbf{y}=\mathrm{o}} = 0. \tag{2.7}$$

Consequently, a Taylor series expansion for h about the origin is

$$h(y_1, y_2, t) = \frac{1}{2}y_1^2\left(\frac{\partial^2 h}{\partial y_1^2}\right)_{\mathbf{y}=\mathrm{o}} + y_1 y_2\left(\frac{\partial^2 h}{\partial y_1\,\partial y_2}\right)_{\mathbf{y}=\mathrm{o}} + \frac{1}{2}y_2^2\left(\frac{\partial^2 h}{\partial y_2^2}\right)_{\mathbf{y}=\mathrm{o}} + \mathcal{O}(r^3), \tag{2.8}$$

where

$$r^2 \equiv y_1^2 + y_2^2. \tag{2.9}$$

Significant simplifications result from introducing the following notation. A Greek suffix can take the values 1 or 2 only. Differentiation with respect to y_α is denoted by, for example,

$$h_\alpha \equiv \frac{\partial h}{\partial y_\alpha}, \qquad h_{\alpha\beta} \equiv \frac{\partial^2 h}{\partial y_\alpha \, \partial y_\beta}. \tag{2.10}$$

And the superscript o (e.g. $h^{\circ}_{\alpha\beta}$) indicates that the quantity is evaluated at the surface point $\mathbf{X}^{\circ}(t)$. Now eqn (2.8) can be written more compactly as

$$h(y_1, y_2, t) = \frac{1}{2} h^{\circ}_{\alpha\beta}(t) y_\alpha y_\beta + \mathcal{O}(r^3). \tag{2.11}$$

(As usual, repeated suffices imply summation.)

The symmetric second-order tensor $h^{\circ}_{\alpha\beta}(t)$ contains information about the curvature of the surface at O: the eigenvalues k_1 and k_2 of $h^{\circ}_{\alpha\beta}$ are the principal curvatures ($k_1 \geq k_2$), and the eigenvectors $\mathbf{e}_1^*$ and $\mathbf{e}_2^*$ are the corresponding principal directions. With y_1^* and y_2^* being coordinates in the $\mathbf{e}_1^*$ and $\mathbf{e}_2^*$ directions, the height function can be written

$$h = \frac{1}{2} k_1 (y_1^*)^2 + \frac{1}{2} k_2 (y_2^*)^2 + \mathcal{O}(r^3). \tag{2.12}$$

As illustrated in Fig. 3, the principal curvatures are (in absolute magnitude) equal to the inverse radii of curvature of the surface. The significance of the signs of k_α is also illustrated on the figure.

The final quantity of interest $S^{\circ}(t)$ measures the amount by which the surface element has been stretched since an initial reference time t_0. The infinitesimal area $\mathrm{d}a(u_{\circ}, v_{\circ}, t)$ of the

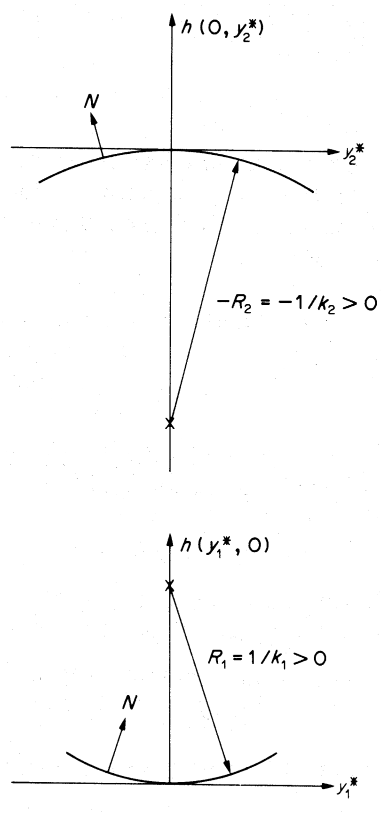

Fig. 3. Sketch of the surface in the neighborhood of O.

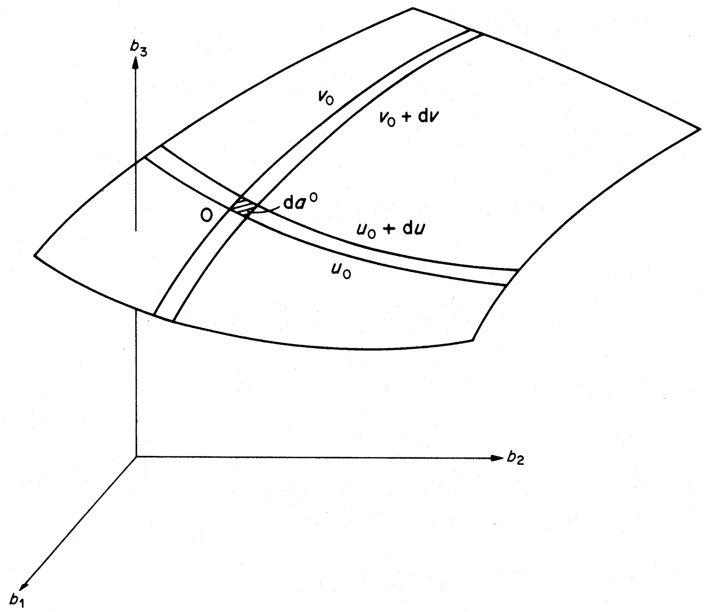

Fig. 4. Sketch of part of the surface showing the infinitesimal area da^o.

surface element sketched on Fig. 4 is

$$da^o(t) = A^o(t)\, du\, dv, \tag{2.13}$$

where

$$A(u, v, t) \equiv \left| \frac{\partial \mathbf{X}}{\partial u} \times \frac{\partial \mathbf{X}}{\partial v} \right|. \tag{2.14}$$

The *stretch factor* $S^o(t)$ defined by

$$S^o(t) \equiv \ln\{A^o(t)/A^o(t_0)\}, \tag{2.15}$$

is the logarithm of the ratio of the surface element areas at times t and t_0. This is an intrinsic property (i.e. independent of the parametrization) and its rate of change—*the stretch rate*—is independent of t_0.

We have now defined all the properties of a surface element. They are: its position $\mathbf{X}^o(t)$; the normal to the surface $\mathbf{N}^o(t)$; the principal curvatures k_1 and k_2; the principal directions $\mathbf{e}_1^*$ and $\mathbf{e}_2^*$; and, the stretch factor $S^o(t)$. All these are intrinsic properties of the surface. They depend on the initial surface element position $\mathbf{X}^o(t_0)$, but are independent of the parametrization and of the choice of $\mathbf{e}_1$ and $\mathbf{e}_2$.

In the next section we discuss the evolution equations for the following properties: $\mathbf{X}^o(t)$, $\mathbf{e}_1(t)$, $\mathbf{e}_2(t)$, $\mathbf{e}_3(t)$, $h^o_{\alpha\beta}(t)$, $k_1(t)$, $k_2(t)$ and $S^o(t)$. These properties are again independent of the parametrization—though $\mathbf{e}_1$, $\mathbf{e}_2$ and $h^o_{\alpha\beta}$ obviously depend upon $\mathbf{e}_1(t)$ and $\mathbf{e}_2(t)$.

2.1. Constant-property surfaces

Some of the surface properties of a constant-property surface can be related to derivatives of the field property $\phi(\mathbf{x}, t)$ at the surface point.

Since ϕ is constant on the surface, $\nabla\phi$ must be normal to it: hence

$$\mathbf{N} = \nabla\phi/|\nabla\phi|. \tag{2.16}$$

In the neighborhood of the surface point $\mathbf{X}^o$, the defining equation [eqn (1.6)] can be rewritten

$$\phi(\mathbf{X}[u, v, t], t) = \phi(\mathbf{X}^o(t) + \mathbf{e}_\alpha(t)y_\alpha + \mathbf{e}_3(t)h(y_1, y_2, t), t) = \phi_0. \tag{2.17}$$

A Taylor series expansion of ϕ about $\mathbf{X}^o$ then yields

$$\phi^o + \phi^o_\alpha y_\alpha + \frac{1}{2}\phi^o_{\alpha\beta} y_\alpha y_\beta + \phi^o_3 h + \mathcal{O}(r^3) = \phi_0. \tag{2.18}$$

Now since

$$\phi^o = \phi_0, \qquad \phi^o_\alpha = 0, \quad \text{and} \quad \phi^o_3 = |\nabla\phi|, \tag{2.19}$$

we obtain

$$h = -\frac{1}{2}\phi^o_{\alpha\beta} y_\alpha y_\beta / |\nabla\phi| + \mathcal{O}(r^3). \tag{2.20}$$

And comparing this result with eqn (2.11), $h^o_{\alpha\beta}$ is given by

$$h^o_{\alpha\beta} = -\phi^o_{\alpha\beta}/|\nabla\phi|. \tag{2.21}$$

The propagation speed w of a constant-property surface [eqn (1.8)] is deduced by differentiating the defining equation with respect to time:

$$\frac{\partial}{\partial t}\phi(\mathbf{X}[u, v, t], t) = 0, \tag{2.22}$$

or

$$\frac{\partial}{\partial t}\phi(\mathbf{x}, t) + \nabla\phi(\mathbf{x}, t)\cdot\frac{\partial}{\partial t}\mathbf{X}(u, v, t) = 0. \tag{2.23}$$

Substituting eqn (2.3) for $\partial\mathbf{X}/\partial t$, and recalling that the normal is $\nabla\phi/|\nabla\phi|$, we obtain the required result:

$$w(u, v, t) = -\left\{\frac{D\phi}{Dt}\Big/|\nabla\phi|\right\}_{\mathbf{X}(u,v,t)}. \tag{2.24}$$

3. EVOLUTION EQUATIONS

The purpose of this section is to present the evolution equations for surface element properties, and to give a simple physical interpretation of each term in these equations. The evolution equation for the surface point location $\mathbf{X}^o$ is stated in the previous section [eqn (2.2)], while the evolution equations for $h^o_{\alpha\beta}$, $\mathbf{e}_i$ and S^o are derived in Appendix A. From these equations, the evolution equations for the principal curvatures k_1 and k_2 are deduced here. It is assumed that the fluid velocity $\mathbf{U}$ and the propagation speed w are twice differentiable functions of position.

The evolution equation for the surface point $\mathbf{X}^o(t)$ [eqn (2.2)] requires little comment, except to note that, for a constant-property surface, as a critical point develops ($|\nabla\phi|^o$ tends to zero) the propagation speed may tend to infinity [eqn (2.24)].

The rotation of the normal to the surface $\mathbf{N}^o = \mathbf{e}_3$ (relative to the fixed B coordinate system) is due to surface gradients of the net propagation speed $w + U_3$. Equation (A. 17) can be rewritten:

$$\dot{\mathbf{e}}_3 = -\mathbf{e}_\alpha\left\{\frac{\partial}{\partial y_\alpha}(w + U_3)\right\}^o, \tag{3.1}$$

where the over-dot denotes the time derivative, and the superscript o indicates that the quantity is evaluated at the surface point (i.e. at $\mathbf{y} = $ o). (It should be noted that $\mathbf{U}$ is the Eulerian velocity—relative to the fixed B coordinate system—even though the components and derivatives are referred to the moving E coordinate system. In addition, $\mathbf{U}$ and its derivatives pertain to the fluid on the positive side of the surface.)

The unit vectors in the tangent plane ($\mathbf{e}_\alpha$, $\alpha = 1, 2$) evolve due to the rotation of $\mathbf{e}_3$, and by the specification that they rotate (in the tangent plane) with the angular velocity of the fluid.

Equation (A. 18) can be rewritten:

$$\dot{\mathbf{e}}_\alpha = \frac{1}{2}\left\{\frac{\partial U_\beta}{\partial y_\alpha} - \frac{\partial U_\alpha}{\partial y_\beta}\right\}^{\circ} \mathbf{e}_\beta + \left\{\frac{\partial}{\partial y_\alpha}(w + U_3)\right\}^{\circ} \mathbf{e}_3. \tag{3.2}$$

Because of this specified rotation, the evolution of $h^{\circ}_{\alpha\beta}$ depends only on the symmetric part of the velocity-gradient tensor. Thus we introduce the rate-of-strain tensor

$$s_{ij} \equiv \frac{1}{2}\left(\frac{\partial U_i}{\partial y_j} + \frac{\partial U_j}{\partial y_i}\right)^{\circ}, \tag{3.3}$$

and denote by s_N and s_T the rates of strain normal to the surface and in the tangent plane:

$$s_N \equiv s_{33} = \left(\frac{\partial U_3}{\partial y_3}\right)^{\circ}, \qquad s_T \equiv s_{\alpha\alpha}. \tag{3.4}$$

The dilatation rate is then

$$\Delta \equiv \left(\frac{\partial U_i}{\partial y_i}\right)^{\circ} = s_T + s_N. \tag{3.5}$$

The symmetric two-dimensional tensor $h^{\circ}_{\alpha\beta}$ contains all the information about the curvature of the surface at the surface point. The evolution equation for $h^{\circ}_{\alpha\beta}$ [eqn (A.27)] can be written:

$$\dot{h}^{\circ}_{\alpha\beta} = \left\{\frac{\partial^2}{\partial y_\alpha\, \partial y_\beta}(w + U_3)\right\}^{\circ} + s_N h^{\circ}_{\alpha\beta} - (s_{\gamma\beta} h^{\circ}_{\alpha\gamma} + s_{\gamma\alpha} h^{\circ}_{\beta\gamma}) + w^{\circ} h^{\circ}_{\alpha\gamma} h^{\circ}_{\gamma\beta}. \tag{3.6}$$

From this equation we deduce the rate of change of the principal curvatures. A superscript * is used to indicate that a quantity is evaluated in the principal axes of $h^{\circ}_{\alpha\beta}$. Thus, in principal axes, $h^{\circ}_{\alpha\beta}$ itself is

$$h^*_{11} = k_1, \qquad h^*_{22} = k_2, \qquad h^*_{12} = h^*_{21} = 0. \tag{3.7}$$

From the eigenvector–eigenvalue equation it can be shown [30] that the rate of change of the eigenvalues at time t is equal to the rate of change of the diagonal components in the fixed coordinate system that at time t coincides with the principal axes. (This result is not obvious since the principal axes change with time.) Thus from eqn (3.6) we obtain

$$\dot{k}_1 = \left\{\frac{\partial^2(w + U_3)}{\partial y^*_1\, \partial y^*_1}\right\}^{\circ} + (s_N - 2s^*_{11})k_1 + w^{\circ} k_1^2. \tag{3.8}$$

Recall that y^*_1 is the coordinate in the principal direction $\mathbf{e}^*_1$. The equation for k_2 is similar.

The first term in eqn (3.8) is the rate of bending of the surface, which is due to second derivative (in the tangent plane) of the net propagation speed $w + U_3$. Since the other terms in the equation are linear or quadratic in k_1, only this first process can bend a plane surface. The bending process is illustrated on Fig. 5 which shows the bending due to $(\partial^2 U_3/\partial y_1^{*2})^{\circ}$ of an initially plane surface. After a time δt, the surface at $y^*_1 = \pm\delta y^*_1$ moves in the y_3 direction by an amount δh:

$$\delta h \approx \frac{1}{2}\left(\frac{\partial^2 U_3}{\partial y_1^{*2}}\right)^{\circ} (\delta y^*_1)^2 \delta t; \tag{3.9}$$

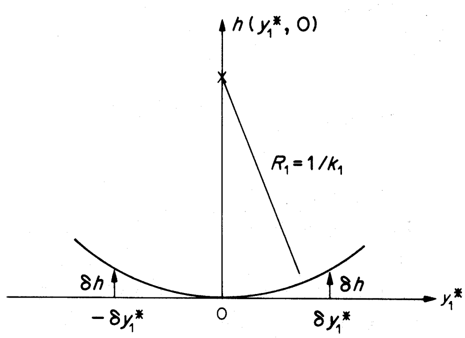

Fig. 5. Sketch showing the effect of bending on an initially plane surface.

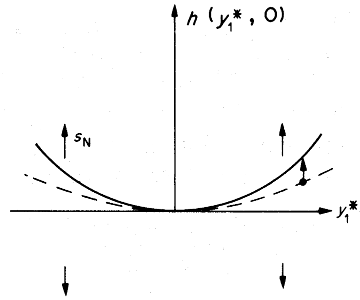

Fig. 6. Sketch showing the effect of positive normal strain to increase curvature.

and the surface in the neighborhood of the origin is a cylinder of radius R_1:

$$R_1^{-1} = k_1 \approx \frac{2\delta h}{(\delta y_1^*)^2} \approx \left(\frac{\partial^2 U_3}{\partial y_1^{*2}}\right)^\circ \delta t. \tag{3.10}$$

In the simplest case of an initially plane material surface in homogeneous isotropic turbulence, the mean $\langle h^\circ_{\alpha\beta}\rangle$ is zero by symmetry. The bending $(\partial^2 U_3/\partial y_\alpha\, \partial y_\beta)^\circ$ is a random process, with zero mean, that scales with the Kolmogorov microscales. Consequently, due to bending alone, the standard deviations of k_1 and k_2 increase initially linearly with t, but ultimately as $t^{1/2}$ (assuming that $(\partial^2 U_3/\partial y_\alpha\, \partial y_\beta)^\circ$ and $h^\circ_{\alpha\beta}$ have finite correlation times).

The second term in eqn (3.8) represents the action of strain to modify existing curvature. Positive strain normal to the surface ($s_N > 0$) causes an increase in the absolute magnitude of k_1, as illustrated in Fig. 6. Conversely, positive strain in the tangent plane ($s^*_{11} > 0$) causes a decrease in $|k_1|$ (Fig. 7). Whether the overall effect of straining is to increase or decrease the curvature is clearly an important question. For homogeneous isotropic turbulence, the question is addressed in Appendix B, but a clear answer is not evident. Straining that is uncorrelated with the curvature tends to decrease curvature, whereas persistent straining increases curvature. A plausible conjecture is that a balance between bending and straining develops, leading to a stationary distribution of curvatures that scales with the Kolmogorov scales.

The final term in eqn (3.8) represents the rate of change of curvature due to the propagation of the surface. Retaining this term alone, eqn (3.8) becomes

$$\dot{k}_1 = w^\circ k_1^2, \tag{3.11}$$

or, in terms of the radius of curvature $R_1 \equiv 1/k_1$,

$$\dot{R}_1 = -w^\circ. \tag{3.12}$$

As illustrated in Fig. 8, the propagation speed w° is also the rate of change of the radius of curvature. The figure shows the case in which k_2 is zero and w° is a positive constant. Then the surface corresponds to an element of a cylinder of radius $|R_1|$ that is expanding (Fig. 8a) or contracting (Fig. 8b) depending upon whether the center of curvature is in the direction of $-\mathbf{N}$ or of $\mathbf{N}$. In the latter case, after a time $R_1(t)/w^\circ$, the radius of curvature becomes zero, and the

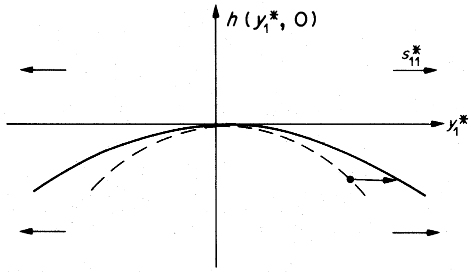

Fig. 7. Sketch showing the effect of positive strain in the tangent plane to decrease curvature.

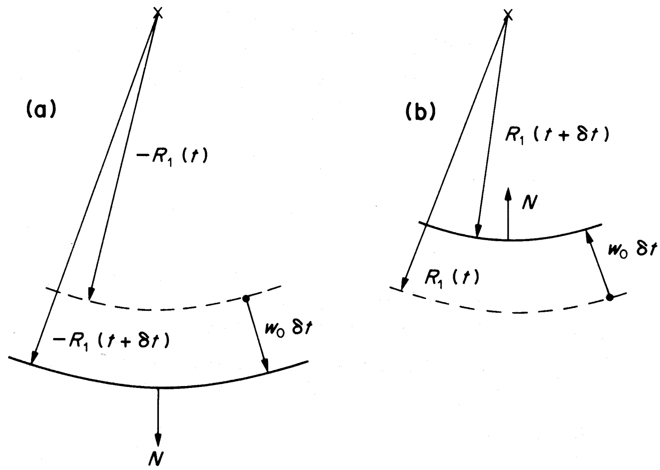

Fig. 8. Sketch showing the effect of propagation on an initially cylindrical surface element with (a) negative curvature, and (b) positive curvature.

principal curvature k_1 becomes infinite. The development of such a singularity is discussed further in the next section.

The *stretching rate* $\dot{S}^\circ$ is the rate of change of the stretch factor S° and measures the fractional rate of increase of surface area at the surface point $\mathbf{X}^\circ$. From eqn (A.15) we obtain

$$\dot{S}^\circ = s_T - w^\circ(k_1 + k_2). \tag{3.13}$$

For a material surface ($w^\circ = 0$), the well-known result is that the fractional rate of change of surface area is equal to the rate of strain in the tangent plane. As conjectured by Batchelor [23] and proved by Cocke [24], in incompressible isotropic turbulence, the mean of s_T is positive. Hence, on average, surface elements are stretched by the turbulent straining.

The second term in eqn (3.13) represents the fractional rate of change of surface area due to the propagation of the surface. For the cases depicted in Figs 8a and 8b, the rates are $w^\circ/|R_1|$ and $-w^\circ/|R_1|$ respectively. In general, the fractional rate of change of area is the same as that for a sphere of radius $R = 2/(k_1 + k_2)$ expanding (or contracting) at the speed $\dot{R} = w^\circ$.

In Appendix A, eqn (3.13) is compared to previous expressions for the stretching rate (or flame stretch), in particular that of Chung and Law [9]. It is shown that all of these expressions are consistent, but that eqn (3.13) has the advantage of separating the effects of straining and curvature in a Gallilean invariant manner.

To close this section, we draw attention to the remarkable fact that for a material surface, or for a propagating surface with constant speed w, the surface element equations are closed with respect to surface properties. By this we mean that, given the velocity field $\mathbf{U}(\mathbf{x}, t)$ and initial conditions for any surface element, the evolution equations for $\mathbf{X}^\circ$, $\mathbf{e}_i$, $h^\circ_{\alpha\beta}$ and S° (eqns (2.2), (3.1), (3.2), (3.6) and (3.13)] form a closed set. Thus each surface element evolves independently. It is not obvious that this should be the case: one might have expected, for example, that the equation for $h^\circ_{\alpha\beta}$ contained third or fourth derivatives of the height function ($h^\circ_{\alpha\beta\gamma}$ or $h^\circ_{\alpha\beta\gamma\delta}$).

4. BREAKDOWN OF REGULARITY

We have assumed the surfaces under consideration to be regular. We now examine how an initially regular surface can cease to be regular because of the development of a singularity, an internal edge, a self-intersection, or a critical point.

4.1 Singularity

The development of a singularity is accompanied by the curvature $|k_1|$ or $|k_2|$ becoming infinite. For a material surface ($w = 0$), eqn (3.8) has the solution

$$k_1(t) = k_1(t_0)I(t_0, t) + \int_{t_0}^{t} I(t', t)\left\{\frac{\partial^2 U_3(t')}{\partial y_1^* \, \partial y_1^*}\right\}^\circ \mathrm{d}t', \tag{4.1}$$

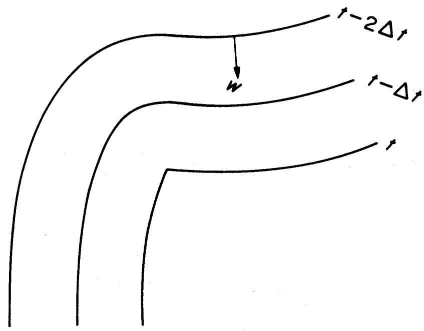

Fig. 9. Sketch to show the formation of a singularity caused by propagation.

where

$$I(t', t) \equiv \exp\left\{\int_{t'}^{t} s_N(t'') - 2s_{11}^*(t'')\, dt''\right\}. \tag{4.2}$$

Since in turbulence the first and second velocity derivatives are bounded it follows that $|k_1|$ (and similarly $|k_2|$) can grow at most exponentially with time. Thus a material surface does not develop a singularity in finite time.

As observed in the previous section, a surface propagating with a constant velocity w can develop a singularity as k_1 tends to $+\infty$: this is illustrated in Fig. 9. There are, however, two mechanisms by which the development of a singularity may be avoided. First, it is possible that the action of turbulent straining could prevent the formation of a singularity—or at least make it improbable. It has been suggested that this occurs in some turbulent premixed flames (see [31]). Second, a singularity may be avoided if w increases with curvature.

To examine this second possibility further, we consider the two-dimensional case ($k_2 = 0$, $\mathbf{e}_2^* = \mathbf{e}_2$) in which the fluid is quiescent ($\mathbf{U} = 0$, everywhere), and in which the propagation speed depends on the curvature k_1. Then eqn (3.8) reduces to

$$\dot{k}_1 = \frac{\partial}{\partial y_1^*}\left(w_k \frac{\partial k_1}{\partial y_1^*}\right) + w^\circ k_1^2, \tag{4.3}$$

where w_k is the rate of change of w with k_1 (evaluated at the surface point). As Sethian [27] has observed, this is a reaction-diffusion equation, and it is reasonable to expect that if w_k is strictly positive a singularity does not develop, because of the smoothing action of the diffusive term. This has not been proved however.

In the extension to the three-dimensional case (but still in quiescent fluid) we suppose that the propagation speed depends on the mean curvative $H \equiv \frac{1}{2}(k_1 + k_2)$. Then, from eqn (3.8) we obtain

$$\dot{H} = \frac{\partial}{\partial y_\alpha^*}\left(w_H \frac{\partial H}{\partial y_\alpha^*}\right) + w^\circ k_\alpha k_\alpha, \tag{4.4}$$

where w_H is the rate of change of w with H. Again, for positive w_H, the diffusive term has a smoothing effect on H. But this is far from proof that a singularity in k_1 or k_2 is avoided.

(We have assumed that the propagation speed w is twice differentiable. If instead w is continuous but not differentiable, then a singularity develops instantly.)

Excluding critical points, a constant-property surface cannot have singularities. This follows directly from eqn (2.21) and the assumption that $\phi(\mathbf{x}, t)$ has bounded second spatial derivatives.

4.2 Internal edges

The way in which an internal edge (i.e. a slit or a hole) might develop is illustrated in Fig. 10. It requires that two initially adjacent surface points (O and P) become separated by a finite

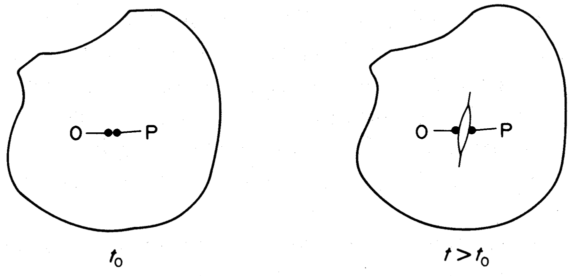

Fig. 10. Sketch illustrating the formation of internal edges as the points O and P separate. (Subject to the assumption made, this cannot occur.)

distance. It is now shown that this cannot happen (if the surface is otherwise regular) and hence slits or holes cannot develop.

The surface point O has position

$$\mathbf{X}^{\mathrm{o}}(t) = \mathbf{X}(u_{\mathrm{o}}, v_{\mathrm{o}}, t), \tag{4.5}$$

while the position of P is

$$\mathbf{X}^{\mathrm{o}}(t) = \mathbf{X}(u_{\mathrm{p}}, v_{\mathrm{p}}, t) = \mathbf{X}^{\mathrm{o}}(t) + \Delta\mathbf{X}(t) = \mathbf{X}^{\mathrm{o}}(t) + \mathbf{e}_i(t)\,\Delta y_i(t). \tag{4.6}$$

This equation defines the separation vector $\Delta\mathbf{X}$ and its coordinates Δy_i in the E system. The evolution equation for Δy_α can be obtained from the equations already derived. With Δr being the separation

$$(\Delta r)^2 \equiv \Delta y_\alpha\,\Delta y_\alpha, \tag{4.7}$$

we obtain

$$\Delta\dot{y}_\alpha = \Delta y_\beta\{s_{\alpha\beta} - w^{\mathrm{o}} h^{\mathrm{o}}_{\alpha\beta}\} + \mathcal{O}(\Delta r^2). \tag{4.8}$$

Provided there are no critical points or singularities, w^{o} and $h^{\mathrm{o}}_{\alpha\beta}$ are finite, as is $s_{\alpha\beta}$. Thus, if the terms of order Δr^2 can be neglected, eqn (4.8) shows that Δr grows at most exponentially with time. Hence for all time $t > t_0$,

$$\lim_{\Delta r(t_0)\to 0} \{\Delta r(t)\} = 0, \tag{4.9}$$

showing that initially adjacent surface points cannot develop a finite separation. Thus the development of an internal edge as depicted on Fig. 10 cannot occur.

(If w varied discontinuously over the surface then internal edges would develop. But, by assumption, w varies continuously.)

4.3 Self-intersections

Figure 11 illustrates the evolution of a surface that leads to a self-intersection. Just as the surfaces touch (Fig. 11b) at time t_i the two points O and P are coincident, and the normals are opposite:

$$\mathbf{X}^{\mathrm{P}}(t_i) = \mathbf{X}^{\mathrm{o}}(t_i), \qquad \mathbf{N}^{\mathrm{P}}(t_i) = -\mathbf{N}^{\mathrm{o}}(t_i). \tag{4.10}$$

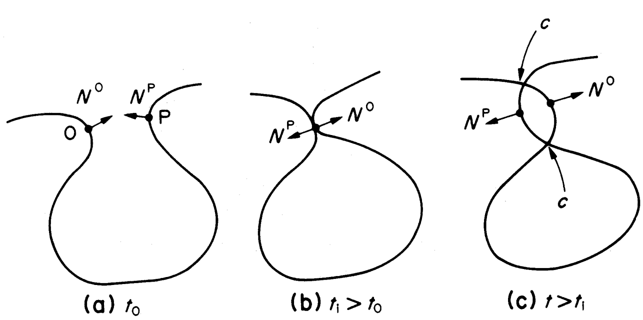

Fig. 11. Sketch illustrating the evolution of a surface leading to a self-intersection at time t_i

A material surface (every point of which is a fluid particle) cannot experience a self-intersection. Let $r(t)$ be the distance between two initially distinct fluid particles ($r(t_0) > 0$). By a method similar to that used in Section 4.2, it is readily shown that $r(t)$ can decrease at most exponentially with time. Thus, the first of eqns (4.10) cannot be satisfied in finite time.

Neither can a constant-property surface experience a self-intersection. The second of eqns (4.10) requires that $(\nabla\phi)^{\circ}$ and $(\nabla\phi)^{\mathrm{P}}$ have opposite signs. But since $\nabla\phi$ is a continuous function, this cannot occur.

A propagating surface obviously can experience self-intersections just as depicted on Fig. 11. In such a case, whether the theory breaks down or not depends upon the physics of the problem. If, physically, part of the surface can pass through another part without affecting its propagation, then the surface equations developed above remain valid through the intersection. But in the simple model of a premixed turbulent flame the intersecting surfaces annihilate each other, and in doing so generate cusps (labelled "*c*" in Fig. 11c). The evolution equations for the local surface properties can neither detect the occurrence of a self-intersection, nor describe the resulting cusps.

4.4 *Critical points*

The parametrization $\mathbf{X}(u, v)$ of the regular surface provides a one-to-one mapping of the surface points in three-dimensional space to the two-dimensional parameter space (u, v). At a critical point no such mapping exists. Crudely, at a critical point the surface is not two-dimensional. A simple—but extreme—example is of a constant-property surface (defined by $\phi(\mathbf{X}) = \phi_0$) in a fluid in which the property $\phi(\mathbf{x})$ has the value ϕ_0 everywhere. Then every point in the fluid is a surface point, and every point is a critical point.

For an initially regular surface, the development of a critical point is accompanied by a breakdown in the parametrization. In terms of surface properties this is indicated by the stretch factor S° eqn (2.15) becoming infinite. The evolution equation for S° [eqn (3.13)] shows that S° remains finite provided that s_{T}, w_{o}, k_1 and k_2 are finite. Thus a regular material or propagating surface cannot develop a critical point.

A constant-property surface has a critical point wherever $\nabla\phi$ is zero. If O is a critical point at time t_0 then a Taylor series expansion of $\phi(\mathbf{x}, t)$ about $(\mathbf{X}^{\circ}, t_0)$ is

$$\phi(\mathbf{X}^{\circ} + \mathbf{b}_i x_i,\, t_0 + \delta t) = \phi^{\circ} + \dot{\phi}^{\circ}\delta t + \frac{1}{2}\,\phi^{\circ}_{ij}x_i x_j \cdots \tag{4.11}$$

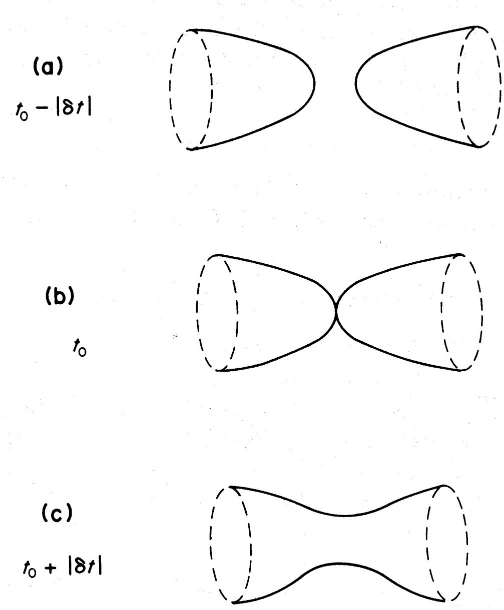

Fig. 12. Sketch showing a critical point in a constant-property surface: $\dot{\phi}^{\circ}$ and one eigenvalue of ϕ°_{ij} positive [eqn (4.12)].

Thus at time $t_0 + \delta t$, the surface points (if any) $\mathbf{X}^o + \mathbf{b}_i Y_i$ in the neighborhood of O satisfy (to first order)

$$\phi^o_{ij} Y_i Y_j = -2\delta t \dot{\phi}^o. \tag{4.12}$$

The nature of the surface near the critical point depends, then, on the eigenvalues ($\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}$) of ϕ^0_{ij}. We briefly describe the normal cases in which $\dot{\phi}^o$ and $\lambda^{(i)}$ are non-zero.

If $\dot{\phi}^o$ and all the eigenvalues are positive, then at time $t_0 - |\delta t|$, the surface is an ellipsoid that contracts to the critical point at time t_0. At time $t_0 + |\delta t|$ there are no surface points in the neighborhood of O. If all eigenvalues are positive but $\dot{\phi}^o$ is negative, then at time $t_0 + |\delta t|$ the surface is an expanding ellipsoid that did not exist prior to t_0. Similar situations arise when all the eigenvalues are negative, but the normal $\mathbf{N}$ points inward rather than outward.

If $\dot{\phi}^o$ and one eigenvalue ($\lambda^{(1)}$, say) are positive and the other eigenvalues are negative, then at time $t_0 - |\delta t|$ the surface consists of two paraboloids (Fig. 12a) that join at time t_0 (Fig. 12b) to form a single tube (Fig. 12c). If $\dot{\phi}^o$ is negative, the reverse of this process occurs.

Even though (in normal cases) the critical point exists just for an instant, the breakdown in the parametrization is permanent.

5. PROBABILISTIC DESCRIPTION

In this section the probabilistic description of the surface at a single fixed point is presented. The extension to many points and to points moving with the surface is straightforward.

First, statistical properties of the surface are defined. These definitions are conceptually straightforward, but is is less straightforward to obtain mathematical expressions for them: this is done in Appendix C. Then, from these mathematical expressions, probabilistic evolution equations are derived (in Appendix C) and discussed.

The *expected surface-to-volume ratio* $\Sigma(\mathbf{x}, t)$ measures the probability of the surface being at $\mathbf{x}$ at time t. Let δV be an infinitesimal volume at $\mathbf{x}$, and let δA be the area of the surface within δV at time t. (δA may well be zero.) Then $\Sigma(\mathbf{x}, t)$ is the expectation of $\delta A/\delta V$. A second physical interpretation of Σ is that its inverse is a striation thickness [2]. If the surface is multiply folded so that locally it appears like parallel planes, then the mean distance between these planes is Σ^{-1}. [An expression for Σ is given by eqns (C.1) and (C.2)].

The unit vectors $\mathbf{e}_i(t)$ and the tensor $h^o_{\alpha\beta}(t)$ describe the geometry of the surface in the neighborhood of the surface point $\mathbf{X}^0(t)$. It is convenient to introduce the direction cosines

$$a_{ij}(t) \equiv \mathbf{b}_i \cdot \mathbf{e}_j(t), \tag{5.1}$$

which are the components of $\mathbf{e}_j$ referred to the fixed B coordinate system.

The joint pdf $f_s(\hat{\mathbb{H}}, \hat{\mathbb{A}}; \mathbf{x}, t)$ is defined to be the probability density of the joint events

$$\mathbb{C} \equiv \{h^o_{\alpha\beta}(t) = \hat{h}_{\alpha\beta},\ a_{ij}(t) = \hat{a}_{ij}\}, \tag{5.2}$$

conditional upon $\mathbf{x}$ being a surface point at time t. The four independent variables $\hat{\mathbb{H}} = \{\hat{h}_{\alpha\beta}\}$ and the nine independent variables $\hat{\mathbb{A}} = \{\hat{a}_{ij}\}$ are sample-space variables corresponding to $h^o_{\alpha\beta}$ and a_{ij}. From this joint pdf, any one-point surface geometry statistic can be determined. For example, at $(\mathbf{x}, t)$, the mean square curvature is

$$\langle M \rangle = \frac{1}{2}(\langle k_1^2 \rangle + \langle k_2^2 \rangle) = \frac{1}{2}\langle h^o_{\alpha\beta} h^o_{\alpha\beta} \rangle = \iint f_s(\hat{\mathbb{H}}, \hat{\mathbb{A}}; \mathbf{x}, t) \frac{1}{2} \hat{h}_{\alpha\beta} \hat{h}_{\alpha\beta} \, d\hat{\mathbb{H}} \, d\hat{\mathbb{A}}, \tag{5.3}$$

where integration is over the whole sample space. (It may be noted that, in view of the relations $h^o_{\alpha\beta} = h^o_{\beta\alpha}$ and $a_{ij} a_{kj} = \delta_{ik}$, the joint pdf contains some redundant information.)

Together Σ and f_s provide a complete one-point statistical description of the surface. The most natural descriptor of the surface is their product

$$F(\hat{\mathbb{H}}, \hat{\mathbb{A}}, \mathbf{x}; t) = \Sigma(\mathbf{x}, t) f_s(\hat{\mathbb{H}}, \hat{\mathbb{A}}; \mathbf{x}, t), \tag{5.4}$$

which is called the *surface density function*. If F is known, Σ and f_s can be recovered by

$$\Sigma(\mathbf{x}, t) = \iint F(\hat{\mathbb{H}}, \hat{\mathbb{A}}, \mathbf{x}; t)\, d\hat{\mathbb{H}}\, d\hat{\mathbb{A}}, \tag{5.5}$$

and

$$f_s(\hat{\mathbb{H}}, \hat{\mathbb{A}}; \mathbf{x}, t) = F(\hat{\mathbb{H}}, \hat{\mathbb{A}}, \mathbf{x}; t)/\Sigma(\mathbf{x}, t). \tag{5.6}$$

An evolution equation for the surface density function F is derived in Appendix C and discussed below. In order to understand this equation we need to define *surface means* and *conditional surface means*. If $Q(\mathbf{x}, t)$ is any field property (e.g. $U_i(\mathbf{x}, t)$), the (unconditional) surface mean $\langle Q(\mathbf{x}, t)\rangle_s$ is the expectation of $Q(\mathbf{x}, t)$ conditional upon $\mathbf{x}$ being a surface point at time t. Similarly if $R(t)$ is a surface property (e.g. $h^{\circ}_{11}(t)$) then $\langle R(t) \mid \mathbf{x}\rangle_s$ is the expectation of $R(t)$ at $\mathbf{x}$ (conditional upon $\mathbf{x}$ being a surface point at time t). Equations (C.7) and (C.8) provide expressions for these unconditional surface means.

The *conditional surface mean* $\langle R(t)\rangle_c$ is the surface mean of $R(t)$ conditional upon the events $\mathbb{C}$ [eqn (5.2)]:

$$\langle R(t)\rangle_c = \langle R(t) \mid \mathbb{H}^{\circ}(t) = \hat{\mathbb{H}}, \hat{\mathbb{A}}^{\circ}(t) = \hat{\mathbb{A}}, \mathbf{x}\rangle_s. \tag{5.7}$$

Equation (C.20) is an expression for $\langle R(t)\rangle_c$.

The evolution equation for the surface density function F is [eqn (C.24)]

$$\frac{\partial F}{\partial t} + \frac{\partial}{\partial x_i}(\langle \dot{X}^{\circ}_i\rangle_c F) + \frac{\partial}{\partial \hat{h}_{\alpha\beta}}(\langle \dot{h}^{\circ}_{\alpha\beta}\rangle_c F) + \frac{\partial}{\partial \hat{a}_{ij}}(\langle \dot{a}^{\circ}_{ij}\rangle_c F) = \langle \dot{S}^{\circ}\rangle_c F. \tag{5.8}$$

The right-hand side corresponds to a source of F due to stretching of the surface. At $(\hat{\mathbb{H}}, \hat{\mathbb{A}}, \mathbf{x})$, the source is linearly proportional to F and to the conditional rate of stretching $\langle \dot{S}^{\circ}\rangle_c$. The terms on the left-hand side are the rate of increase of F and divergences of fluxes in $\mathbf{x}$, $\hat{\mathbb{H}}$ and $\hat{\mathbb{A}}$ spaces. The fluxes are proportional to F and to the conditional rates of change of the corresponding surface properties.

The evolution equation for the expected surface-to-volume ratio Σ is obtained by integrating eqn (5.8) over all $\hat{\mathbb{H}}$ and $\hat{\mathbb{A}}$: the result [eqn (C.30)] is

$$\frac{\partial \Sigma}{\partial t} + \frac{\partial}{\partial x_i}(\langle \dot{X}^{\circ}_i \mid \mathbf{x}\rangle_s \Sigma) = \langle \dot{S}^{\circ} \mid \mathbf{x}\rangle_s \Sigma. \tag{5.9}$$

Thus there is a source of Σ that is linearly proportional to Σ and to the (unconditional) surface mean stretching rate. In physical space, the flux of Σ is proportional to Σ and to the (unconditional) surface mean velocity of the surface.

As written, the evolution equations for F and Σ [eqns (5.8) and (5.9)] contain no physics: they are mathematical deductions from the definitions of these quantities. The physics enters when the conditional means of the rates of change are evaluated from the deterministic evolution equations for the surface properties. This is done in Appendix C [eqns (C.25)–(C.28)]. The important conclusion is that the conditional means are functions of the independent variables $\hat{\mathbb{H}}$ and $\hat{\mathbb{A}}$ and of the conditional surface means of $\mathbf{U}$, w and their first and second derivatives.

Consider a surface propagating with a fixed velocity W or a material surface (i.e. $W = 0$). For this case we have

$$\langle w\rangle_c = W, \qquad \left\langle \frac{\partial w}{\partial x_i}\right\rangle_c = \left\langle \frac{\partial^2 w}{\partial x_i\, \partial x_j}\right\rangle_c = 0. \tag{5.10}$$

Then the only unknowns in the evolution equation for F are $\langle \mathbf{U}\rangle_c$, $\langle \partial U_i/\partial x_j\rangle_c$ and $\langle \partial^2 U_i/\partial x_j\, \partial x_k\rangle_c$. Consequently only these velocity-field statistics are required in order to solve the equation for F.

Consider, further, a propagating surface (e.g. a premixed flame), the propagation speed of which depends arbitrarily on the surface curvature and linearly on the rate of strain in the tangent plane. This dependence can be written

$$w^{\circ} = W_1(H^{\circ}, M^{\circ}) + W_2(H^{\circ}, M^{\circ}) s_{\mathrm{T}}, \tag{5.11}$$

where

$$H^{\circ} \equiv h^{\circ}_{\alpha\alpha}, \qquad M^{\circ} \equiv \frac{1}{2} h^{\circ}_{\alpha\beta} h^{\circ}_{\alpha\beta}, \tag{5.12}$$

and W_1 and W_2 are given functions. For this case, the conditional surface mean of w is

$$\langle w \rangle_c = W_1(\hat{H}, \hat{M}) + W_2(\hat{H}, \hat{M}) \hat{a}_{i\alpha} \hat{a}_{j\alpha} \langle \partial U_i / \partial x_j \rangle_c, \tag{5.13}$$

where

$$\hat{H} \equiv \hat{h}_{\alpha\alpha} \quad \text{and} \quad \hat{M} \equiv \frac{1}{2} \hat{h}_{\alpha\beta} \hat{h}_{\alpha\beta}. \tag{5.14}$$

Thus $\langle w \rangle_c$ is known in terms of the independent variables, W_1, W_2 and $\langle \partial U_i / \partial x_j \rangle_c$. To this extent the variation of w with curvature and strain can be incorporated without introducing further unknowns. The surface variations of w (e.g. the term $\langle \partial w / \partial x_i \rangle_c \hat{a}_{i\alpha}$) cannot be incorporated so simply. But in some circumstances their neglect may be justifiable.

The surface density function $F(\hat{\mathbb{H}}, \hat{\mathbb{A}}, \mathbf{x}; t)$ is a function of a large number of independent variables—17 for the general case. If the conditional means were known (via modelling, say) the evolution equation for F could, in principle, be solved. But because of the large dimensionality, conventional numerical methods (e.g. finite-differences) are impracticable. This is precisely the same problem faced in pdf methods (see, for example, [19]), and the same solution is available: the evolution equation can be solved by a Monte Carlo method.

It is beyond the scope of this paper to describe the Monte Carlo method and associated modelling, and to prove its convergence to the evolution equation for F. We note however that Monte Carlo solutions to the equation for F are quite feasible: several solutions to pdf equations of large dimensionality have been obtained (see [19] for references). In the Monte Carlo method for F, the principal quantities to be modelled are the velocity and its first two derivatives following a surface element. Pope and Cheng [37] have applied the method to turbulent premixed flames, using models based on the Langevin equation [19, 32, 33].

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APPENDIX A

Derivation of Deterministic Evolution Equations

In deriving the evolution equations for the local surface properties, our starting points are the evolution equation for the surface point location:

$$\frac{\partial \mathbf{X}}{\partial t}(u, v, t) = \mathbf{U}(\mathbf{X}[u, v, t], t) + w(u, v, t)\mathbf{N}(u, v, t); \tag{A.1}$$

the representation of $\mathbf{X}$ in the B and E coordinate systems:

$$\mathbf{X}(u, v, t) = \mathbf{b}_i X_i^{\circ}(t) + \mathbf{e}_i(t) y_i(u, v, t); \tag{A.2}$$

and, the definition of the height function

$$h(y_1[u, v, t], y_2[u, v, t], t) = y_3(u, v, t). \tag{A.3}$$

In order to simplify the notation in the subsequent derivations, the arguments (u, v, t, etc.) are generally omitted, a superscript o denotes quantities evaluated at the surface point $\mathbf{X}^{\circ}(t) = \mathbf{X}(u_{\circ}, v_{\circ}, t)$, and an over-dot (e.g. $\dot{\mathbf{X}}$) denotes a time derivative.

Stretch factor

The vector $\mathbf{A}(u, v, t)$ is defined by

$$\mathbf{A} = \mathbf{X}_u \times \mathbf{X}_v, \tag{A.4}$$

where the subscripts denote differentiation with respect to u and v. The scalar $A(u, v, t)$ is the modulus of $\mathbf{A}$, and the normal to the surface is

$$\mathbf{N} = \mathbf{A}/A. \tag{A.5}$$

The stretch factor is defined by

$$S(u, v, t) \equiv \ln\{A(u, v, t)/A(u, v, t_0)\}. \tag{A.6}$$

Differentiating eqn (A.6) with respect to time we obtain

$$\begin{aligned} \dot{S} &= \dot{A}/A = \mathbf{A} \cdot \dot{\mathbf{A}}/A^2 \\ &= \mathbf{N} \cdot (\dot{\mathbf{X}}_u \times \mathbf{X}_v + \mathbf{X}_u \times \dot{\mathbf{X}}_v)/A. \end{aligned} \tag{A.7}$$

An expression for $\dot{\mathbf{X}}_u$ is obtained by differentiating eqn (A.1) with respect to u:

$$\dot{\mathbf{X}}_u = \mathbf{X}_u \cdot \nabla \mathbf{U} + \mathbf{N}[\mathbf{X}_u \cdot \nabla w] + w\mathbf{N}_u, \tag{A.8}$$

and similarly for $\dot{\mathbf{X}}_v$. When this expression is substituted into eqn (A.7), the term in $\mathbf{N}$ makes no contribution since $\mathbf{N} \cdot (\mathbf{N} \times \mathbf{X}_v)$ is zero. Thus we obtain

$$\begin{aligned} \dot{S} &= \mathbf{N} \cdot ([\mathbf{X}_u \cdot \nabla \mathbf{U}] \times \mathbf{X}_v + \mathbf{X}_u \times [\mathbf{X}_v \cdot \nabla \mathbf{U}])/A \\ &\quad + w\mathbf{N} \cdot (\mathbf{N}_u \times \mathbf{X}_v + \mathbf{X}_u \times \mathbf{N}_v)/A. \end{aligned} \tag{A.9}$$

The evaluation of eqn (A.9) is facilitated by the observation that the left-hand side is an intrinsic function (i.e. independent of the parametrization), while the right-hand side contains no time derivatives. Thus, the right-hand side can be evaluated with any parametrization, the simplest being (y_1, y_2) in place of (u, v). In this case we obtain, from eqn (A.2),

$$\mathbf{X} = \mathbf{X}^{\circ} + \mathbf{e}_{\alpha} y_{\alpha} + \mathbf{e}_3 h, \tag{A.10}$$

and then,

$$\mathbf{X}_\alpha = \mathbf{e}_\alpha + \mathbf{e}_3 h_\alpha, \tag{A.11}$$

where the subscript α (except in $\mathbf{e}_\alpha$) indicates the differential with respect to y_α. The normal is

$$\mathbf{N} = (-\mathbf{e}_\alpha h_\alpha + \mathbf{e}_3)/(1 + h_\beta h_\beta)^{1/2}, \tag{A.12}$$

(where the denominator is A), and its derivative with respect to y_α is

$$\mathbf{N}_\alpha = -\mathbf{e}_\beta h_{\alpha\beta}/A - \mathbf{N} h_{\alpha\beta} h_\beta / A^2. \tag{A.13}$$

At the surface point $\mathbf{X}^\circ(t)$, h°_α is zero and so we obtain, simply,

$$\mathbf{X}^\circ_\alpha = \mathbf{e}_\alpha, \qquad \mathbf{N}^\circ = \mathbf{e}_3, \qquad A^\circ = 1, \qquad \mathbf{N}^\circ_\alpha = -\mathbf{e}_\beta h^\circ_{\alpha\beta}. \tag{A.14}$$

With these expressions, evaluating eqn (A.9) at the surface point $\mathbf{X}^\circ(t)$ yields:

$$\dot{S}^\circ = U^\circ_{\alpha,\alpha} - w^\circ h^\circ_{\alpha\alpha}. \tag{A.15}$$

In eqn (A.15) it should be noted that $\mathbf{U}$ is the Eulerian velocity—relative to the fixed B-coordinate system—even though the component and derivative are referred to the moving E-coordinate system. This equation is compared to previous expressions for flame stretch at the end of this appendix.

Normal to the surface

Differentiating eqn (A.5) yields

$$\begin{aligned}\dot{\mathbf{N}} &= \dot{\mathbf{A}}/A - \mathbf{N}\dot{S} \\ &= (\dot{\mathbf{X}}_u \times \mathbf{X}_v + \mathbf{X}_u \times \dot{\mathbf{X}}_v)/A - \mathbf{N}\dot{S}.\end{aligned} \tag{A.16}$$

Once eqns (A.8) and (A.9) have been used to substitute for $\dot{\mathbf{X}}_u$, $\dot{\mathbf{X}}_v$ and $\dot{S}$, the right-hand side is an intrinsic property that contains no time derivatives. Thus, as before, it can be evaluated with (y_1, y_2) replacing the (u, v) parametrization. At the surface point $\mathbf{X}^\circ$ this yields:

$$\dot{\mathbf{N}}^\circ = -\mathbf{e}_\alpha (U^\circ_{3,\alpha} + w^\circ_\alpha). \tag{A.17}$$

Rotation of the E-coordinate system

The unit vector $\mathbf{e}_3$, being identical to $\mathbf{N}^\circ$, evolves according to eqn (A.17). The requirement that $\mathbf{e}_i \cdot \mathbf{e}_j$ does not change with time, then shows that the $\mathbf{e}_3$ component of $\dot{\mathbf{e}}_\alpha$ must be $(U^\circ_{3,\alpha} + w^\circ_\alpha)$. In order to aid the interpretation of the equations, the unit vectors in the tangent plane ($\mathbf{e}_1$ and $\mathbf{e}_2$) are made to rotate with the angular velocity of the fluid in the tangent plane. This yields

$$\dot{\mathbf{e}}_\alpha = \frac{1}{2}\mathbf{e}_\beta (U^\circ_{\beta,\alpha} - U^\circ_{\alpha,\beta}) + \mathbf{e}_3 (U^\circ_{3,\alpha} + w^\circ_\alpha). \tag{A.18}$$

Height function

Differentiating eqn (A.3) with respect to time (with u and v fixed) yields

$$\dot{h} + h_\alpha \dot{y}_\alpha = \dot{y}_3. \tag{A.19}$$

Note that $\dot{h}$ is the rate of change of h at fixed y_1 and y_2, while $\dot{y}_i$ is the rate of change of y_i at fixed u and v. An evolution equation for y_j is obtained by differentiating eqn (A.2) with respect to time, taking the dot product with $\mathbf{e}_j$, and then using eqn (A.1) to substitute for $\dot{\mathbf{X}}$:

$$\dot{y}_j = -\mathbf{e}_j \cdot \dot{\mathbf{e}}_i y_i + (U_j - U^\circ_j) + (wN_j - w^\circ N^\circ_j). \tag{A.20}$$

We are interested in h only in the immediate neighborhood of $\mathbf{X}^\circ$. Specifically, since h° and h°_α are zero, we are interested in the second derivative $h^\circ_{\alpha\beta}$ that contains the information about the local curvature. Expanding $\dot{h}$ in a Taylor series about $y_\alpha = 0$, we obtain

$$\dot{h} = \frac{1}{2}\dot{h}^\circ_{\alpha\beta} y_\alpha y_\beta + \mathcal{O}(r^3), \tag{A.21}$$

where $r^2 = y_\alpha y_\alpha$. Thus $\dot{h}^\circ_{\alpha\beta}$ can be obtained by determining $\dot{h}$ to second order in r from Eqn (A.19). This, in turn, requires that [from eqn (A.20)] $\dot{y}_3$ be determined to second order and that $\dot{y}_\alpha$ be determined to first order (because h_α is of order r). This is achieved by expanding the terms in brackets in eqn (A.20) in Taylor series. For $\dot{y}_\alpha$ we obtain:

$$\dot{y}_\alpha = \frac{1}{2}(U^\circ_{\alpha,\beta} + U^\circ_{\beta,\alpha}) y_\beta - w^\circ h^\circ_{\alpha\beta} y_\beta + \mathcal{O}(r^2); \tag{A.22}$$

and for $\dot{y}_3$:

$$\dot{y}_3 = \frac{1}{2} y_\alpha y_\beta \{U^\circ_{3,\alpha\beta} + w^\circ_{\alpha\beta} + U^\circ_{3,3} h^\circ_{\alpha\beta} - w^\circ h^\circ_{\alpha\gamma} h^\circ_{\gamma\beta}\} + \mathcal{O}(r^3). \tag{A.23}$$

Substituting the last two results into eqn. (A.19) yields:

$$\dot{h} = \frac{1}{2} y_\alpha y_\beta \{s_{33} h^\circ_{\alpha\beta} - 2 s_{\gamma\beta} h^\circ_{\alpha\gamma} + U^\circ_{3,\alpha\beta} + w^\circ_{\alpha\beta} + w^\circ h^\circ_{\alpha\gamma} h^\circ_{\gamma\beta}\} + \mathcal{O}(r^3), \tag{A.24}$$

where $s_{\alpha\beta}$ are the components of the rate-of-strain tensor

$$s_{\alpha\beta} = \frac{1}{2}(U^{\circ}_{\alpha,\beta} + U^{\circ}_{\beta,\alpha}). \tag{A.25}$$

In eqn (A.24), all the terms in braces are symmetric except for that containing $s_{\alpha\beta}$. This term can be rewritten

$$y_{\alpha}y_{\beta}\{-s_{\gamma\beta}h^{\circ}_{\alpha\gamma}\} = \frac{1}{2}y_{\alpha}y_{\beta}\{-(s_{\gamma\beta}h^{\circ}_{\alpha\gamma} + s_{\gamma\alpha}h^{\circ}_{\beta\gamma})\}. \tag{A.26}$$

Then, comparing eqn (A.24) with eqn (A.21) we deduce that $h^{\circ}_{\alpha\beta}$ evolves by

$$\dot{h}^{\circ}_{\alpha\beta} = s_{33}h^{\circ}_{\alpha\beta} - (s_{\gamma\beta}h^{\circ}_{\alpha\gamma} + s_{\gamma\alpha}h^{\circ}_{\beta\gamma}) + U^{\circ}_{3,\alpha\beta} + w^{\circ}_{\alpha\beta} + w^{\circ}h^{\circ}_{\alpha\gamma}h^{\circ}_{\gamma\beta}. \tag{A.27}$$

Equations in fixed coordinates

The evolution of the surface properties is best understood in the moving E coordinate system: but for some purposes (e.g. those of Section 5) we need to express the evolution equations in the fixed B coordinate system. The direction cosines $a_{ij}(t)$ are defined by

$$a_{ij}(t) = \mathbf{b}_i \cdot \mathbf{e}_j(t), \tag{A.28}$$

and hence they evolve according to

$$\dot{a}_{ij}(t) = \mathbf{b}_i \cdot \dot{\mathbf{e}}_j(t). \tag{A.29}$$

Then the velocity gradients can be transformed by

$$U_{\alpha,\beta} = \left(\frac{\partial U_i}{\partial x_j}\right)^{\circ} a_{i\alpha}a_{j\beta}, \tag{A.30}$$

for example. It should be understood that, on the left-hand side the components of $U^{\circ}_{\alpha,\beta}$ pertain to the E-system while, on the right-hand side the components of $\partial U_i/\partial x_j$ pertain to the B-system.

The best way to express gradients of the propagation speed w depends upon the case considered. We suppose w to be defined for all $\mathbf{x}$ and hence obtain

$$w^{\circ}_{\alpha\beta} = \left(\frac{\partial^2 w}{\partial x_i\,\partial x_j}\right)^{\circ} a_{i\alpha}a_{j\beta}, \tag{A.31}$$

for example.

With these transformations, the complete set of surface evolution equations is:

$$\dot{X}^{\circ}_i = U^{\circ}_i + w^{\circ}a_{i3}, \tag{A.32}$$

$$\dot{S}^{\circ} = \left(\frac{\partial U_i}{\partial x_j}\right)^{\circ} a_{i\alpha}a_{j\alpha} - w^{\circ}h^{\circ}_{\alpha\alpha}, \tag{A.33}$$

$$\dot{h}^{\circ}_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)^{\circ}\{a_{i3}a_{j3}h^{\circ}_{\alpha\beta} - a_{i\gamma}(h^{\circ}_{\alpha\gamma}a_{j\beta} + h^{\circ}_{\beta\gamma}a_{j\alpha})\} + \left\{\left(\frac{\partial^2 U_i}{\partial x_j\,\partial x_k}\right)^{\circ} a_{i3} + \frac{\partial^2 w}{\partial x_j\,\partial x_k}\right\}a_{j\alpha}a_{k\beta} + w^{\circ}h^{\circ}_{\alpha\gamma}h^{\circ}_{\gamma\beta}, \tag{A.34}$$

$$\dot{a}_{i\alpha} = \left(\frac{\partial U_j}{\partial x_k}\right)^{\circ}\left\{\frac{1}{2}a_{i\beta}(a_{j\beta}a_{k\alpha} - a_{j\alpha}a_{k\beta}) + a_{i3}a_{j3}a_{k\alpha}\right\} + \left(\frac{\partial w}{\partial x_j}\right)^{\circ} a_{i3}a_{j\alpha}, \tag{A.35}$$

and

$$\dot{a}_{i3} = -a_{i\alpha}\left\{\left(\frac{\partial U_j}{\partial x_k}\right)^{\circ} a_{j3}a_{k\alpha} + \left(\frac{\partial w}{\partial x_j}\right)^{\circ} a_{j\alpha}\right\}. \tag{A.36}$$

Again, note that differentiation and the components of $\mathbf{U}^{\circ}$ are referred to the B-coordinate system.

Comparison of expressions for stretching rate

In the context of premixed laminar flames, several different expressions for the stretching rate—or flame stretch—have been reported [8, 9, 34]. Matalon demonstrated that his expression is consistent with Buckmaster's while Chung and Law demonstrated that theirs is consistent with Matalon's. Hence it suffices here to show that the present expression [eqn (3.13) or (A.15)] is consistent with Chung and Law's. But it is also observed that the present expression has the advantage over the previous formulae of separating the effects of straining and curvature in a Gallilean invariant manner.

Chung and Law's expression for flame stretch can be written

$$\dot{S} = \nabla_T \cdot \mathbf{V}_T + (\mathbf{V}\cdot\mathbf{N})\nabla_T\cdot\mathbf{N}, \tag{A.37}$$

where $\mathbf{V}$ is the total velocity of a surface point,

$$\mathbf{V} \equiv \mathbf{U} + w\mathbf{N}, \tag{A.38}$$

$\mathbf{V}_T$ is the velocity in the local tangent plane

$$\mathbf{V}_T \equiv \mathbf{V} - \mathbf{N}(\mathbf{V}\cdot\mathbf{N}), \tag{A.39}$$

and ∇_T is the two-dimensional gradient operator in the local tangent plane. Using the following relations:

$$\mathbf{V}_T = \mathbf{U}_T \equiv \mathbf{U} - \mathbf{N}(\mathbf{U}\cdot\mathbf{N}), \tag{A.40}$$

$$\nabla_T\cdot\mathbf{N} = -(k_1 + k_2), \tag{A.41}$$

and

$$s_T = \nabla_T \cdot \mathbf{U}, \tag{A.42}$$

Chung and Law's expression [eqn (A.37)] can be rewritten:

$$\dot{S} = \nabla_T \cdot \mathbf{U}_T - (w + \mathbf{U} \cdot \mathbf{N})(k_1 + k_2), \tag{A.43}$$

while eqn (A.15) becomes:

$$\dot{S} = \nabla_T \cdot \mathbf{U} - w(k_1 + k_2). \tag{A.44}$$

The equivalence of eqns (A.43) and (A.44) is readily demonstrated. From eqns (A.40) and (A.41) we have

$$\begin{aligned} \nabla_T \cdot \mathbf{U}_T &= \nabla_T \cdot \mathbf{U} - (\nabla_T \cdot \mathbf{N})(\mathbf{U} \cdot \mathbf{N}) \\ &= \nabla_T \cdot \mathbf{U} + \mathbf{U} \cdot \mathbf{N}(k_1 + k_2). \end{aligned} \tag{A.45}$$

Thus, substituting eqn (A.45) into eqn (A.43) we obtain eqn (A.44).

Both eqns (A.43) and (A.44) appear to separate the stretching rate into a straining in the tangent plane and a term due to the mean curvature $(k_1 + k_2)$. But the second expression is to be preferred since the two terms on the right-hand side of Eqn (A.44) are invariant under Gallilean transformations, while those in eqn (A.43) are not. Thus the separation of the straining and curvature effects in eqn (A.44) is independent of the motion of the coordinate system.

APPENDIX B

Effect of Strain on Curvature

We consider the effect of strain on the curvature of a material surface in homogeneous, isotropic, incompressible turbulence. The magnitude of the curvature is characterized by the mean-square curvature

$$M = \frac{1}{2}(k_1^2 + k_2^2) = \frac{1}{2} h^{\circ}_{\alpha\beta} h^{\circ}_{\alpha\beta}. \tag{B.1}$$

It may be seen from eqn (3.6) or (3.8) that strain could cause the principal curvatures (and hence M) to increase exponentially with time. The question addressed here is: does strain cause the curvature to increase without bound, or does strain tend (on average) to decrease curvature? If the latter is the case then a balance between bending and straining would be established leading to a distribution of curvatures centered, presumably, on the inverse of the Kolmogorov length scale.

The effect of straining *alone* on $h^{\circ}_{\alpha\beta}$ is [eqn (3.6)]

$$\dot{h}^{\circ}_{\alpha\beta} = -s_T h^{\circ}_{\alpha\beta} - (s_{\gamma\beta} h^{\circ}_{\alpha\gamma} + s_{\gamma\alpha} h^{\circ}_{\beta\gamma}), \tag{B.2}$$

where the continuity equation

$$s_N + s_T = \Delta = 0, \tag{B.3}$$

has been used to eliminate s_N.

Uncorrelated straining

The rate-of-strain in the tangent plane can be decomposed into isotropic and deviatoric parts

$$s_{\alpha\beta} = \frac{1}{2} s_T \delta_{\alpha\beta} + s'_{\alpha\beta}. \tag{B.4}$$

Equation (B.2) can now be written

$$\dot{h}^{\circ}_{\alpha\beta} = -2s_T h^{\circ}_{\alpha\beta} - (s'_{\gamma\beta} h^{\circ}_{\alpha\gamma} + s'_{\gamma\alpha} h^{\circ}_{\beta\gamma}), \tag{B.5}$$

and hence we obtain for the mean-square curvature

$$\dot{M} = -2s_T M - (s'_{\gamma\beta} h^{\circ}_{\alpha\gamma} h^{\circ}_{\alpha\beta} + s'_{\gamma\alpha} h^{\circ}_{\beta\gamma} h^{\circ}_{\alpha\beta}). \tag{B.6}$$

Now, on taking the mean of this equation, if the final term could be neglected (on the ground that $s'_{\alpha\beta}$ and $h^{\circ}_{\alpha\gamma} h^{\circ}_{\gamma\beta}$ are uncorrelated), then we obtain

$$\frac{\partial \langle \ln M \rangle}{\partial t} = -2\langle s_T \rangle. \tag{B.7}$$

Since it has been proved [24, 25] that $\langle s_T \rangle$ is positive, it follows that uncorrelated straining tends to decrease curvature.

Persistent straining

Since $s'_{\alpha\beta}$ and $h^{\circ}_{\alpha\beta}$ are (for the case considered) isotropic and $s'_{\alpha\beta}$ has zero mean, these tensors will be uncorrelated if their principal axes are independent. But we show now that persistent straining tends to align the principal axes.

Consider the case in which $s_{\alpha\beta}$ and s_T are constant, and $\mathbf{e}_1(t_0)$ and $\mathbf{e}_2(t_0)$ are chosen to be the principal directions of $s_{\alpha\beta}$. Then in the E coordinate system $s_{\alpha\beta}$ is

$$\begin{bmatrix} \frac{1}{2}s_T + s'_T & 0 \\ 0 & \frac{1}{2}s_T - s'_T \end{bmatrix}$$

where

$$s'_T = s'_{11} \equiv \frac{1}{2}(s_{11} - s_{22}). \tag{B.8}$$

The effect of straining alone on $h^\circ_{\alpha\beta}$ is then [eqn (B.5)]:

$$\frac{d}{dt}\begin{bmatrix} h^\circ_{11} & h^\circ_{12} \\ h^\circ_{21} & h^\circ_{22} \end{bmatrix} = -2s_T\begin{bmatrix} h^\circ_{11} & h^\circ_{12} \\ h^\circ_{21} & h^\circ_{22} \end{bmatrix} + 2\begin{bmatrix} -h^\circ_{11}s'_T & 0 \\ 0 & s'_T h^\circ_{22} \end{bmatrix}, \tag{B.9}$$

to which the solution is

$$\begin{bmatrix} h^\circ_{11}(t) & h^\circ_{12}(t) \\ h^\circ_{21}(t) & h^\circ_{22}(t) \end{bmatrix} = \exp(-2s_T[t-t_0])\begin{bmatrix} h^\circ_{11}(t_0)e^{-2s'_T(t-t_0)} & h^\circ_{12}(t_0) \\ h^\circ_{21}(t_0) & h^\circ_{22}(t_0)e^{2s'_T(t-t_0)} \end{bmatrix}. \tag{B.10}$$

Consider the case in which $h^\circ_{11}(t_0)$ is positive and s'_T is negative. Then $h^\circ_{11}(t)$ increases exponentially with time compared to the other components, and hence $\mathbf{e}^*_1$ becomes aligned with $\mathbf{e}_1$. In general, the eigenvector of $h^\circ_{\alpha\beta}(\mathbf{e}^*_1$ or $\mathbf{e}^*_2)$ corresponding to the eigenvalue of greatest absolute magnitude (k_1 or k_2), tends to become aligned with the principal direction of the smallest principal strain in the tangent plane.

It may readily be seen from eqn (B.10) that the mean-square curvature increases exponentially if $|s'_T|$ is greater than s_T.

The above considerations show that the fate of the mean-square curvature depends upon the alignment of the principal axes of strain and curvature. The question whether M increases without bound is unanswered.

Gauss curvature

Another measure of curvature is the Gauss curvature

$$K \equiv k_1 k_2. \tag{B.11}$$

For the case considered, from eqn (3.8) we obtain

$$\dot{K} = -4s_T K + k_2\left(\frac{\partial^2 U_3}{\partial y^*_1\,\partial y^*_1}\right)^\circ + k_1\left(\frac{\partial^2 U_3}{\partial y^*_2\,\partial y^*_2}\right)^\circ. \tag{B.12}$$

Since on average s_T is positive, the effect of strain is clearly to tend to decrease $\ln|K|$. But, for two reasons, this observation is insufficient to show that the principal curvatures remain bounded. First, k_1 (say) can become infinite while K tends to zero. [This occurs with persistent straining and $s'_T < s_T < 0$, see eqn (B.9).] Second, if $|k_1|$ or $|k_2|$ becomes large, the effect of straining can be overwhelmed by the effects of bending [the final two terms in eqn (B.12)].

APPENDIX C

Probabilistic Evolution Equations

In this appendix, several one-point surface statistics are defined and related to the parametrized description of the surface. These statistics are: the expected surface-to-volume ratio Σ; the surface mean $\langle Q\rangle_s$ of a property Q; the joint pdf f_s of the surface properties $\mathbb{H}^\circ$ and $\mathbb{A}$; the surface density function F; and, the conditional surface mean $\langle Q\rangle_c$. Then the evolution equations for F and Σ are derived.

Expected surface-to-volume ratio

It is shown that the expected surface-to-volume ratio $\Sigma(\mathbf{x}, t)$ is given by

$$\Sigma(\mathbf{x}, t) = \langle \Sigma'(\mathbf{x}, t)\rangle, \tag{C.1}$$

where the fine-grained surface-to-volume ratio $\Sigma'(\mathbf{x}, t)$ is defined by

$$\Sigma'(\mathbf{x}, t) = \iint_{\mathcal{U}} \delta(\mathbf{x} - \mathbf{X}[u, v, t])A(u, v, t)\,du\,dv. \tag{C.2}$$

Here $\delta(\mathbf{x})$ denotes the delta-function product $\delta(x_1)\delta(x_2)\delta(x_3)$, $A(u, v, t)$ is given by eqn (2.14), and $\mathcal{U}$ is the region in the parameter space corresponding to all points on the surface $\mathcal{S}(t)$.

Consider now a simple volume V in physical space. Let $\mathcal{S}_v(t)$ denote the part of the surface that lies within V, let $A_v(t)$ be its area, and let $\mathcal{U}_v(t)$ denote the region in the parameter space corresponding to points on $\mathcal{S}_v(t)$. Integrating eqn (C.2) over V yields

$$\iiint_V \Sigma'(\mathbf{x}, t)\,d\mathbf{x} = \iint_{\mathcal{U}}\left\{\iiint_V \delta(\mathbf{x} - \mathbf{X}[u, v, t])\,d\mathbf{x}\right\}A(u, v, t)\,du\,dv, \tag{C.3}$$

(where $d\mathbf{x}$ is written for $dx_1\,dx_2\,dx_3$). Now if the point $\mathbf{X}[u, v, t]$ lies within V the term in braces is unity, otherwise it is zero. Thus, dividing by V, eqn (C.3) becomes

$$\frac{1}{V}\iiint_V \Sigma'(\mathbf{x}, t)\,d\mathbf{x} = \frac{1}{V}\iint_{\mathcal{U}_v} A(u, v, t)\,du\,dv = A_v(t)/V. \tag{C.4}$$

The left-hand side is the volume-average of Σ', while the right-hand side is the surface-to-volume ratio. Since this equation is valid for any choice of V, Σ' must be the local surface-to-volume ratio.

[By way of example, if the surface is the $x_1 - x_2$ plane (i.e. $X_3 = 0$), then from eqn (C.3) we obtain $\Sigma'(\mathbf{x}, t) = \delta(x_3)$.] Thus, as stated in eqn (C.1), the expected surface-to-volume ratio $\Sigma(\mathbf{x}, t)$ is the expectation of $\Sigma'(\mathbf{x}, t)$.

Surface means

If $Q(\mathbf{x}, t)$ is any field property (e.g. $U_i(\mathbf{x}, t)$), we define the *surface mean* $\langle Q(\mathbf{x}, t)\rangle_s$ to be the mean of Q conditional upon $\mathbf{x}$ being a surface point at time t. In order to obtain an expression for $\langle Q(\mathbf{x}, t)\rangle_s$ consider the quantity

$$Q^*(\mathbf{x}, t) \equiv \iint_{\mathcal{U}} Q(\mathbf{x}, t)\delta(\mathbf{x} - \mathbf{X}[u, v, t])\, \mathrm{d}u\, \mathrm{d}v. \tag{C.5}$$

Integrating eqn (C.5) over the volume V yields

$$\begin{aligned}\iiint_V Q^*(\mathbf{x}, t)\, \mathrm{d}\mathbf{x} &= \iint_{\mathcal{U}} \left\{\iiint_V Q(\mathbf{x}, t)\delta(\mathbf{x} - \mathbf{X}[u, v, t])\, \mathrm{d}\mathbf{x}\right\} A(u, v, t)\, \mathrm{d}u\, \mathrm{d}v \\ &= \iint_{\mathcal{U}} \left\{\iiint_V \delta(\mathbf{x} - \mathbf{X}[u, v, t])\, \mathrm{d}\mathbf{x}\right\} Q(\mathbf{X}[u, v, t], t)A(u, v, t)\, \mathrm{d}u\, \mathrm{d}v \\ &= \iint_{\mathcal{U}_v} Q(\mathbf{X}[u, v, t], t)A(u, v, t)\, \mathrm{d}u\, \mathrm{d}v. \end{aligned} \tag{C.6}$$

The last expression is the surface integral of Q over the part of the surface $\mathcal{S}_v(t)$ within the volume V. Thus the quantity

$$\iiint_V \langle Q^*(\mathbf{x}, t)\rangle\, \mathrm{d}\mathbf{x} \Big/ \iiint_V \Sigma(\mathbf{x}, t)\, \mathrm{d}\mathbf{x},$$

is the expected surface integral of Q divided by the expected surface area; or, in other words, the surface average of the expectation of Q. Since the volume V can be chosen arbitrarily, $\langle Q^*(\mathbf{x}, t)\rangle/\Sigma(\mathbf{x}, t)$ must be the local surface mean Q. Hence from eqn (C.5) we obtain the required expression for the unconditional surface mean:

$$\langle Q(\mathbf{x}, t)\rangle_s = \iint_{\mathcal{U}} \langle Q(\mathbf{x}, t)\delta(\mathbf{x} - \mathbf{X}[u, v, t])A(u, v, t)\rangle \mathrm{d}u\, \mathrm{d}v/\Sigma(\mathbf{x}, t). \tag{C.7}$$

[Of course, if $\Sigma(x, t)$ is zero, $\langle Q(\mathbf{x}, t)\rangle_s$ is undefined.]

A similar analysis can be performed to obtain an expression for the surface mean $\langle R(t) \mid \mathbf{x}\rangle_s$ of a surface property $R(u, v, t)$. This again is the mean of R conditional upon $\mathbf{x}$ being a surface point at time t. The result equivalent to eqn (C.7) is

$$\begin{aligned}\langle R(t) \mid \mathbf{x}_s\rangle &= \iint_{\mathcal{U}} \langle R(u, v, t)\delta(\mathbf{x} - \mathbf{X}[u, v, t])A(u, v, t)\rangle\, \mathrm{d}u\, \mathrm{d}v/\Sigma(\mathbf{x}, t). \\ &= \iint_{\mathcal{U}} \langle R^{\circ}(t)\delta(\mathbf{x} - \mathbf{X}^{\circ}(t))A^{\circ}(t)\rangle \mathrm{d}u_{\mathrm{o}}\, \mathrm{d}v_{\mathrm{o}}/\Sigma(\mathbf{x}, t). \end{aligned} \tag{C.8}$$

The final expression is obtained simply by replacing the integration variables u and v by u_{o} and v_{o}. [Recall that the superscript o indicates that the quantity is evaluated at the surface point $\mathbf{X}^{\circ}(t) = \mathbf{X}(u_{\mathrm{o}}, v_{\mathrm{o}}, t)$.]

Joint pdf of surface properties

The joint pdf $f_s(\hat{\mathbb{H}}, \hat{\mathbb{A}}; \mathbf{x}, t)$ is defined to be the probability density of the joint events $\mathbb{C}$:

$$\mathbb{C} \equiv \{h^{\circ}_{\alpha\beta}(t) = \hat{h}_{\alpha\beta},\, a_{ij}(t) = \hat{a}_{ij}\}, \tag{C.9}$$

conditional upon $\mathbf{x}$ being a surface point at time t (i.e. conditional upon $\mathbf{X}^{\circ}(t) = \mathbf{x}$).

An expression for f_s can be obtained as the surface mean of a delta-function product. If ϕ is a random variable, then its pdf $f_\phi(\psi)$ can be written

$$f_\phi(\psi) = \langle \delta(\psi - \phi)\rangle, \tag{C.10}$$

[19, 35, 36]. Similarly, taking the surface mean [eqn (C.8)] of the delta-function product

$$\delta(\hat{\mathbb{H}} - \mathbb{H}^{\circ}[t])\delta(\hat{\mathbb{A}} - \mathbb{A}[t]) \equiv \prod_{\substack{\alpha,\beta=1,2 \\ i,j=1,3}} \delta(\hat{h}_{\alpha\beta} - h^{\circ}_{\alpha\beta}[t])\delta(\hat{a}_{ij} - a_{ij}[t]), \tag{C.11}$$

we obtain [from eqn (C.8)]

$$\begin{aligned} f_s(\hat{\mathbb{H}}, \hat{\mathbb{A}}; \mathbf{x}, t) &= \langle \delta(\hat{\mathbb{H}} - \mathbb{H}^{\circ}[t])\delta(\hat{\mathbb{A}} - \mathbb{A}[t]) \mid \mathbf{x}\rangle_s \\ &= \iint_{\mathcal{U}} \langle \delta(\hat{\mathbb{H}} - \mathbb{H}^{\circ}[t])\delta(\mathbb{A} - \mathbb{A}[t])\delta(\mathbf{x} - \mathbf{X}^{\circ}[t])A^{\circ}(t)\rangle \mathrm{d}u_{\mathrm{o}}\, \mathrm{d}v_{\mathrm{o}}/\Sigma(\mathbf{x}, t). \end{aligned} \tag{C.12}$$

To abbreviate the notation we introduce

$$G(\hat{\mathbb{H}}, \hat{\mathbb{A}}, \mathbf{x}, u_{\mathrm{o}}, v_{\mathrm{o}}, t) \equiv \delta(\hat{\mathbb{H}} - \mathbb{H}^{\circ}[t])\delta(\hat{\mathbb{A}} - \mathbb{A}[t])\delta(\mathbf{x} - \mathbf{X}^{\circ}[t])A^{\circ}(t). \tag{C.13}$$

Then the joint pdf of surface properties can be written

$$f_s = \iint_{\mathscr{U}} \langle G \rangle \, du_o \, dv_o / \Sigma(\mathbf{x}, t). \tag{C.14}$$

Here and below, the arguments of G are those given in eqn (C.13).

Surface density function

Expressions for the expected surface-to-volume ratio $\Sigma(\mathbf{x}, t)$ and the joint pdf of surface properties $f_s(\hat{\mathbb{H}}, \hat{\mathbb{A}}; \mathbf{x}, t)$ have now been obtained [eqns (C.1), (C.2) and (C.12), (C.14)]. Together, Σ and f_s provide a complete one-point statistical description of the surface. The same information is contained in the *surface density function* defined by

$$F(\hat{\mathbb{H}}, \hat{\mathbb{A}}, \mathbf{x}; t) \equiv \iint_{\mathscr{U}} \langle G \rangle \, du_o \, dv_0. \tag{C.15}$$

From this definition it may be seen that $\Sigma(\mathbf{x}, t)$ is obtained by integrating over all $\hat{\mathbb{A}}$ and $\hat{\mathbb{H}}$

$$\iint F(\hat{\mathbb{H}}, \hat{\mathbb{A}}, \mathbf{x}; t) \, d\hat{\mathbb{H}} \, d\hat{\mathbb{A}} = \iint_{\mathscr{U}} \langle \delta(\mathbf{x} - \mathbf{X}^o[t]) A^o(t) \rangle \, du_o \, dv_o = \Sigma(\mathbf{x}, t), \tag{C.16}$$

[cf. eqns (C.1) and (C.2)]. Then, from eqn (C.14), f_s is given by

$$f_s(\hat{\mathbb{H}}, \hat{\mathbb{A}}; \mathbf{x}, t) = F(\hat{\mathbb{H}}, \hat{\mathbb{A}}, \mathbf{x}; t) / \Sigma(\mathbf{x}, t). \tag{C.17}$$

Conditional surface means

Let $R^o(t)$ be a surface property (i.e. $R^o(t) = R(u_o, v_o, t)$) or a field property (i.e. $R^o(t) = R(\mathbf{X}^o[t], t)$. We seek an expression for the surface mean of $R^o(t)$ conditional upon the events $\mathbb{C}$ [i.e. $\hat{\mathbb{H}} = \mathbb{H}^o, \hat{\mathbb{A}} = \mathbb{A}$, eqn (C.9)].

Now let $f_R(\hat{R}, \hat{\mathbb{H}}, \hat{\mathbb{A}}; \mathbf{x}, t)$ be the joint pdf of $R^o(t)$, $\mathbb{H}^o(t)$ and $\mathbb{A}(t)$, conditional upon $\mathbf{x}$ being a surface point at time t. Then the pdf of $R^o(t)$ conditional upon $\mathbb{C}$ is

$$f_c(\hat{R}; \mathbf{x}, t \mid \hat{\mathbb{H}}, \hat{\mathbb{A}}) = f_R(\hat{R}, \hat{\mathbb{H}}, \hat{\mathbb{A}}; \mathbf{x}, t) / f_s(\hat{\mathbb{H}}, \hat{\mathbb{A}}; \mathbf{x}, t). \tag{C.18}$$

And the required conditional surface mean of $R^o(t)$ is

$$\langle R^o(t) \rangle_c \equiv \langle R^o(t) \mid \hat{\mathbb{H}}, \hat{\mathbb{A}}, \mathbf{x} \rangle_s = \int f_c(\hat{R}; \mathbf{x}, t \mid \hat{\mathbb{H}}, \hat{\mathbb{A}}) \hat{R} \, d\hat{R}. \tag{C.19}$$

Writing f_R in terms of delta functions [similar to eqn (C.12)] and then substituting eqn (C.18) into eqn (C.19) we obtain:

$$\begin{aligned}
\langle R^o(t) \rangle_c &= \int \hat{R} \Big\{ \iint_{\mathscr{U}} \langle \delta(\hat{R} - R^o[t]) \delta(\hat{\mathbb{H}} - \mathbb{H}^o[t]) \delta(\hat{\mathbb{A}} - \mathbb{A}[t]) \\
&\quad \times \delta(\mathbf{x} - \mathbf{X}^o[t]) A^o(t) \rangle \, du_o \, dv_o / \Sigma(\mathbf{x}, t) \Big\} \Big/ f_s(\hat{\mathbb{H}}, \hat{\mathbb{A}}; \mathbf{x}, t) \, d\hat{R} \\
&= \iint_{\mathscr{U}} \langle R^o[t] \delta(\hat{\mathbb{H}} - \mathbb{H}^o[t]) \delta(\hat{\mathbb{A}} - \mathbb{A}[t]) \\
&\quad \times \delta(\mathbf{x} - \mathbf{X}^o[t]) A^o(t) \rangle \, du_o \, dv_o / F(\hat{\mathbb{H}}, \hat{\mathbb{A}}, \mathbf{x}, t) \\
&= \iint_{\mathscr{U}} \langle R^o(t) G \rangle \, du_o \, dv_o / F,
\end{aligned} \tag{C.20}$$

which is the required result.

Evolution equation for F

We proceed now to derive the evolution equation for the surface density function. The derivation is similar to that used for joint pdf equations (see [19] for example). The method is to differentiate the defining equation for F[eqn (C.15)] with respect to time. The derivatives of the delta functions can be written, for example,

$$\begin{aligned}
\frac{\partial}{\partial t} \delta(\hat{\mathbb{A}} - \mathbb{A}) &= -\frac{\partial a_{ij}}{\partial t} \frac{\partial}{\partial \hat{a}_{ij}} \delta(\hat{\mathbb{A}} - \mathbb{A}) \\
&= -\frac{\partial}{\partial \hat{a}_{ij}} \{ \dot{a}_{ij} \delta(\hat{\mathbb{A}} - \mathbb{A}) \},
\end{aligned} \tag{C.21}$$

where the summation convention applies as usual. (Note that $\mathbb{A}$ is independent of $\hat{\mathbb{A}}$ which is independent of t.) Thus differentiating eqn (C.15) with G defined by eqn (C.13) we obtain

$$\frac{\partial F}{\partial t} = \iint_{\mathscr{U}} \left\langle -\frac{\partial}{\partial \hat{h}_{\alpha\beta}} \{ \dot{h}^o_{\alpha\beta} G \} - \frac{\partial}{\partial \hat{a}_{ij}} \{ \dot{a}_{ij} G \} - \frac{\partial}{\partial x_i} \{ \dot{X}_i G \} + \frac{\dot{A}^o}{A^o} G \right\rangle du_o \, dv_o. \tag{C.22}$$

The term in $\dot{h}^{\circ}_{\alpha\beta}$ can be re-expressed as:

$$\iint_{\mathcal{U}} \left\langle -\frac{\partial}{\partial \hat{h}_{\alpha\beta}} \{\dot{h}^{\circ}_{\alpha\beta} G\} \right\rangle \mathrm{d}u_{\circ}\, \mathrm{d}v_{\circ} = -\frac{\partial}{\partial \hat{h}_{\alpha\beta}} \iint_{\mathcal{U}} \langle \dot{h}^{\circ}_{\alpha\beta} G \rangle \, \mathrm{d}u_{\circ}\, \mathrm{d}v_{\circ} = -\frac{\partial}{\partial \hat{h}_{\alpha\beta}} (\langle \dot{h}^{\circ}_{\alpha\beta} \rangle_c F), \tag{C.23}$$

the last line following from the expression for the conditional expectation, eqn (C.20). And recalling that $\dot{A}^{\circ}/A^{\circ}$ is the rate of stretching $\dot{S}^{\circ}$, eqn (C.22) becomes

$$\frac{\partial F}{\partial t} + \frac{\partial}{\partial x_i}(\langle \dot{X}^{\circ}_i \rangle_c F) + \frac{\partial}{\partial \hat{h}_{\alpha\beta}}(\langle \dot{h}^{\circ}_{\alpha\beta} \rangle_c F) + \frac{\partial}{\partial \hat{a}_{ij}}(\langle \dot{a}_{ij} \rangle_c F) = \langle \dot{S}^{\circ} \rangle_c F. \tag{C.24}$$

Expressions for the conditional expectations are obtained by taking the conditional expectations of eqns (A.32)–(A.36):

$$\langle \dot{X}^{\circ}_i \rangle_c = \langle U_i \rangle_c + \langle w \rangle_c \hat{a}_{i3}, \tag{C.25}$$

$$\langle \dot{h}^{\circ}_{\alpha\beta} \rangle_c = \frac{1}{2} \left\langle \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right\rangle_c \{\hat{a}_{i3}\hat{a}_{j3}\hat{h}_{\alpha\beta} - \hat{a}_{i\gamma}(\hat{h}_{\alpha\gamma}\hat{a}_{j\beta} + \hat{h}_{\beta\gamma}\hat{a}_{j\alpha})\}$$
$$+ \left\{ \left\langle \frac{\partial^2 U_i}{\partial x_j\, \partial x_k} \right\rangle_c \hat{a}_{i3} + \left\langle \frac{\partial^2 w}{\partial x_j\, \partial x_k} \right\rangle_c \right\} \hat{a}_{j\alpha}\hat{a}_{k\beta} + \langle w \rangle_c \hat{h}_{\alpha\gamma}\hat{h}_{\gamma\beta}, \tag{C.26}$$

$$\langle \dot{a}_{i\alpha} \rangle_c = \left\langle \frac{\partial U_j}{\partial x_k} \right\rangle_c \left\{ \frac{1}{2} \hat{a}_{i\beta}(\hat{a}_{j\beta}\hat{a}_{k\alpha} - \hat{a}_{j\alpha}\hat{a}_{k\beta}) + \hat{a}_{i3}\hat{a}_{j3}\hat{a}_{k\alpha} \right\} + \left\langle \frac{\partial w}{\partial x_j} \right\rangle_c \hat{a}_{i3}\hat{a}_{j\alpha}, \tag{C.27}$$

$$\langle \dot{a}_{i3} \rangle_c = -\left\langle \frac{\partial U_j}{\partial x_k} \right\rangle_c \hat{a}_{i\alpha}\hat{a}_{j3}\hat{a}_{k\alpha} - \left\langle \frac{\partial w}{\partial x_j} \right\rangle_c \hat{a}_{i\alpha}\hat{a}_{j\alpha},$$

and

$$\langle \dot{S}^{\circ} \rangle_c = \left\langle \frac{\partial U_i}{\partial x_j} \right\rangle_c \hat{a}_{i\alpha}\hat{a}_{j\alpha} - \langle w \rangle_c \hat{h}_{\alpha\alpha}. \tag{C.28}$$

It may be noted that the only conditional expectations in these equations are those of **U**, w and their first and second derivatives.

Evolution equation for Σ

Since the expected surface-to-volume ratio $\Sigma(\mathbf{x}, t)$ is the integral of $F(\hat{\mathbb{H}}, \hat{\mathbb{A}}, \mathbf{x}; t)$ over all $\hat{\mathbb{H}}$ and $\hat{\mathbb{A}}$ [eqn (C.16)], the evolution equation Σ can be obtained by integrating that for F [eqn (C.24)]. In this process, the terms in $\dot{h}^{\circ}_{\alpha\beta}$ and $\dot{a}_{ij}$ vanish. For the term in $\dot{X}^{\circ}_i$ we have:

$$\iint \langle \dot{X}^{\circ}_i \rangle_c F \, \mathrm{d}\hat{\mathbb{H}}\, \mathrm{d}\hat{\mathbb{A}} = \iint \langle \dot{X}^{\circ}_i(t) \mid \hat{\mathbb{H}}, \hat{\mathbb{A}}, \mathbf{x} \rangle f_s(\hat{\mathbb{H}}, \hat{\mathbb{A}}; \mathbf{x}, t) \Sigma(\mathbf{x}, t)\, \mathrm{d}\hat{\mathbb{H}}\, \mathrm{d}\hat{\mathbb{A}}$$
$$= \Sigma(\mathbf{x}, t) \langle \dot{X}^{\circ}_i(t) \mid \mathbf{x} \rangle_s. \tag{C.29}$$

With a similar treatment of the term in $\dot{S}^{\circ}$, from eqn (C.24) we obtain

$$\frac{\partial \Sigma}{\partial t} + \frac{\partial}{\partial x_i}(\langle \dot{X}^{\circ}_i \mid \mathbf{x} \rangle_s \Sigma) = \langle \dot{S}^{\circ} \mid \mathbf{x} \rangle_s \Sigma. \tag{C.30}$$