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Analysis of the conditionally cubic-Gaussian stochastic Lagrangian model

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Abstract
The basis for the conditionally cubic-Gaussian (CCG) stochastic Lagrangian model (Lamorgese *et al* 2007 *J. Fluid Mech.* **582** 423) is briefly reviewed and its large-time behaviour further addressed. To this end, we perform additional multiple-scales calculations which support the adiabatic elimination result of our previous analysis. Lagrangian intermittency in the CCG model is briefly addressed and found to be consistent with the findings of previous works.

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1. Introduction

Recently, intermittency and non-Gaussian effects of fluid particle acceleration (and their Reynolds-number dependences) have been addressed in the conditionally cubic-Gaussian (CCG) stochastic Lagrangian model [1]. In the formulation of this model systematic procedures are employed for incorporating non-Gaussian one-time statistics and intermittent two-time statistics (and their Reynolds-number dependences) from direct numerical simulations (DNS) of homogeneous turbulence with Taylor-scale Reynolds numbers up to 650 on a 2048^3 grid. Adiabatic elimination [2] is one such procedure which has been shown useful [1] for removing dissipative range information from the CCG model and for analyzing the large-time behaviour of two-time velocity statistics predicted by the model. In this paper, we briefly review the basis for the CCG model and further investigate the adiabatic elimination result of our previous analysis [1]. Finally, we briefly address inertial-range behaviour in the CCG model, by showing model predictions for Lagrangian velocity increment PDFs and Lagrangian velocity structure functions against DNS.

2. CCG stochastic Lagrangian model

The CCG model is a stochastic model for the fluid particle acceleration $A(t)$ and it also involves the fluid particle velocity $U(t)$ and a conditioning variable $\chi(t)$. The latter is taken to be $\chi(t) \equiv \ln \varphi(t)/\langle\varphi\rangle$, where φ denotes the pseudo-dissipation. A Lagrangian DNS database for homogeneous turbulence has been interrogated to determine the joint-statistical behaviour of those quantities (see [3, 4]). It is found that the one-time distribution of χ is close to Gaussian with variance σ_χ^2 and its autocorrelation is close to exponential (with timescale T_χ). This supports a modelling assumption for $\chi(t)$ as an Ornstein–Uhlenbeck process [1, 5].

The conditional variance of acceleration $\sigma_{A|\varphi}^2 = \langle A^2|\varphi\rangle$ accounts for the major effects of intermittency of dissipation on acceleration [6]. Based on the DNS data [3, 6], an empirical expression

$$\mathcal{S}^2 \equiv \frac{\sigma_{A|\varphi}^2}{a_\eta^2} = \frac{1.2}{R_\lambda^{0.2}}\left(\frac{\varphi}{\langle\varphi\rangle}\right)^{0.15} + \ln\left(\frac{R_\lambda}{20}\right)\left(\frac{\varphi}{\langle\varphi\rangle}\right)^{1.25}, \qquad (1)$$

(where $a_\eta = (\langle\varphi\rangle^3/\nu)^{1/4}$, with ν the fluid kinematic viscosity, and $R_\lambda = \sqrt{\frac{15\sigma_U^4}{\nu\langle\varphi\rangle}}$ is the Taylor-scale Reynolds number, σ_U being the velocity standard deviation) is utilized in the CCG to accurately describe the variation of $\sigma_{A|\varphi}^2$ with φ and with Reynolds number, in a way that deviates from the Kolmogorov (1962) prediction [1].

When considering the joint-statistics of acceleration and pseudo-dissipation [3, 6], a significant quantity is the conditionally standardized acceleration $\tilde{A} = A/\sigma_{A|\varphi}$. Its conditional mean and variance are zero and one, respectively, and the DNS data [6] show the one-time PDF of $\tilde{A}|\varphi$ as very nearly universal, and, in particular, cubic-Gaussian [1, 6]. In other words, given $\varphi = \hat{\varphi}$ (where $\hat{\varphi}$ is arbitrary) the acceleration $\tilde{A}$ can be modelled as cubic-Gaussian, i.e.

$$\tilde{A} = C[(1-p)\bar{A} + p\bar{A}^3], \tag{2}$$

where $\bar{A}$ is a standardized Gaussian random variable and C is determined by the standardization condition $\langle \tilde{A}^2 \rangle = 1$ as $C(p) = (1+4p+10p^2)^{-1/2}$. A value of $p \approx 0.1$ results from the observation that the conditional flatness $\mu_4(\tilde{A}|\hat{\varphi}) \approx 8$ (approximately independent of $\hat{\varphi}$ and the Reynolds number) [1, 6].

Based on the observations from DNS, the CCG model is by construction exactly consistent with a stationary one-time distribution g of $(U, \bar{A}, \chi)$ in which U, $\bar{A}$ and χ are independent Gaussian variables [1], i.e.

$$g = \frac{1}{\sigma_U\sqrt{2\pi}} \exp\left(-\frac{v^2}{2\sigma_U^2}\right) \frac{1}{\sqrt{2\pi}} \times \exp\left(-\frac{\bar{a}^2}{2}\right) \frac{1}{\sigma_\chi\sqrt{2\pi}} \exp\left(-\frac{x^{*2}}{2\sigma_\chi^2}\right), \tag{3}$$

where v, $\bar{a}$ and x^* denote sample-space variables for U, $\bar{A}$ and $\chi^* \equiv \chi - \langle\chi\rangle$. The imposition of this PDF leads to the CCG equations [1]:

$$\mathrm{d}U = A\mathrm{d}t = \sigma_{A|\varphi}(c_1\bar{A} + c_2\bar{A}^3)\,\mathrm{d}t, \tag{4}$$

$$\mathrm{d}\bar{A} = -\frac{\bar{b}^2}{2}\bar{A}\,\mathrm{d}t - \frac{\sigma_{A|\varphi}}{\sigma_U^2}U(c_1 + 2c_2 + c_2\bar{A}^2)\,\mathrm{d}t + \bar{b}\,\mathrm{d}W, \tag{5}$$

$$\mathrm{d}\chi^* = -\chi^*\frac{\mathrm{d}t}{T_\chi} + \sqrt{\frac{2\sigma_\chi^2}{T_\chi}}\,\mathrm{d}W', \tag{6}$$

where $c_1 = C(1-p)$ and $c_2 = Cp$. In equations (5) and (6), $\bar{b}$ is a diffusion coefficient, and W and W' independent Wiener processes. Being a second-order model, the CCG model accounts for Reynolds-number effects in a natural way. The Reynolds-number dependence for $\sigma_{A|\varphi}/a_\eta$ is specified by equation (1). Furthermore, based on the DNS data [1, 3, 6], $T_\chi/T = 0.055 + 3.55R_\lambda^{0.7}$ (where $T \equiv 1.5\sigma_U^2/\langle\varphi\rangle$). As mentioned above, p in equation (2) is approximately independent of the Reynolds number so that c_1 and c_2 are constants in equations (4) and (5). The Reynolds-number dependence of $\tau_\eta\bar{b}^2$ is unknown but can be investigated [1] by means of the technique of adiabatic elimination [2], as discussed further below.

3. Specification of diffusion coefficient

The CCG model consists of three stochastic differential equations (equations (4)–(6)) with the coefficients σ_χ^2, T_χ, $\sigma_{A|\varphi}^2$ and $\bar{b}$. The timescale $T \equiv 1.5\sigma_U^2/\langle\varphi\rangle$ characterizes the velocity $U(t)$, whereas the Kolmogorov temporal microscale $\tau_\eta \equiv \sqrt{\nu/\langle\varphi\rangle}$ is appropriate for describing the evolution of $A(t)$. At high Reynolds number, T is widely separated from τ_η and A is a 'fast' variable compared to U. This limit corresponds to the adiabatic elimination of acceleration from the stochastic model [1]. An approximate multiple-scales analysis [1] suggests that, in that limit, $U(t)$ and $\chi(t)$ evolve by

$$\mathrm{d}U = -\frac{U}{\mathcal{T}_{L,\varphi}}\,\mathrm{d}t + \sqrt{\frac{2\sigma_U^2}{\mathcal{T}_{L,\varphi}}}\,\mathrm{d}W, \tag{7}$$

$$\mathrm{d}\chi = -\frac{\chi}{T_\chi}\,\mathrm{d}t + \sqrt{\frac{2\sigma_\chi^2}{T_\chi}}\,\mathrm{d}W'. \tag{8}$$

In the appendices, additional multiple-scales calculations are performed to further investigate the velocity conditional timescale $\mathcal{T}_{L,\varphi}(\chi)$.

For the sake of clarity, we recall definitions of dimensionless variables [1]

$$\tau = \frac{a_\eta}{\sigma_U}t, \quad \bar{v} = \frac{v}{\sigma_U}, \quad \bar{x} = \frac{x^*}{\sigma_\chi}, \tag{9}$$

along with the small parameter $\epsilon \equiv \frac{u_\eta}{\sigma_U} = \frac{15^{1/4}}{\sqrt{R_\lambda}}$ (with $u_\eta \equiv (\nu\langle\varphi\rangle)^{1/4}$). Then, the Fokker–Planck equation for the one-time joint PDF $f(\bar{v}, \bar{a}, \bar{x}; \tau)$ associated with equations (4)–(6) can be written as [1]

$$\epsilon\left\{\frac{\partial}{\partial\tau} + \underbrace{\frac{\sigma_{A|\varphi}}{a_\eta}}_{\mathcal{S}}\left[(c_1\bar{a} + c_2\bar{a}^3)\frac{\partial}{\partial\bar{v}} - 2c_2\bar{v}\bar{a} - \bar{v}(c_3 + c_2\bar{a}^2)\frac{\partial}{\partial\bar{a}}\right]\right\} f = \underbrace{\frac{\tau_\eta\bar{b}^2}{2}}_{\mathcal{B}}\mathcal{L}_{\bar{a}}f + \tfrac{2}{3}\epsilon^2 h\mathcal{L}_{\bar{x}}f, \tag{10}$$

where $h \equiv T/T_\chi$, $\mathcal{L}_y \equiv \frac{\partial}{\partial y}(y + \frac{\partial}{\partial y})$ and $c_3 = c_1 + 2c_2$. The ϵ-dependences for $\mathcal{S}$ and h follow from the Reynolds-number dependences noted above. However, the ϵ-dependence for $\mathcal{B} \equiv \frac{1}{2}\tau_\eta\bar{b}^2$ is unknown. Multiple-scales calculations are shown in appendix C that allow for Reynolds-number dependences in the coefficients (previous calculations [1] are also shown in appendix B for clarity). Two analyses in the appendices are based on different assumptions or approximations concerning $\mathcal{S}$, $\mathcal{B}$ and h but lead to the same conclusion [1] that adiabatic elimination of acceleration from the CCG model yields a velocity-dissipation model, equations (7) and (8), with the conditional velocity timescale $\mathcal{T}_{L,\varphi}$ given by

$$\frac{\mathcal{T}_{L,\varphi}}{T} = \frac{\tau_\eta\bar{b}^2}{3\delta(\sigma_{A|\varphi}a_\eta)^2}, \tag{11}$$

where $\delta = C^2(1+4p+6p^2)$. In [1] this result has been coupled with the observation that the velocity-dissipation model is exactly solvable and can be matched with DNS data for second-order conditional Lagrangian velocity structure functions to back out φ-dependences for $\tau_\eta\bar{b}^2$ at different Reynolds numbers. In other words, the diffusion coefficient for the CCG model can be specified in terms of the conditional velocity timescale as follows [1]:

$$\frac{T}{\mathcal{T}_{L,\varphi}} = \alpha + \beta\left(\frac{\varphi}{\langle\varepsilon\rangle}\right)^{1/2}, \tag{12}$$

$$\alpha = 2.9, \quad \beta = \beta_0\sqrt{R_\lambda}, \quad \beta_0 = 0.16. \tag{13}$$

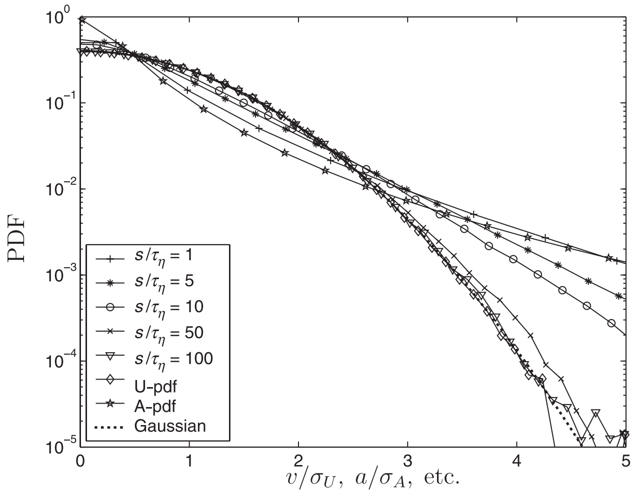

Figure 1. CCG predictions for $R_\lambda = 650$ based on equations (12) and (13) for standardized PDFs of Lagrangian velocity increments.

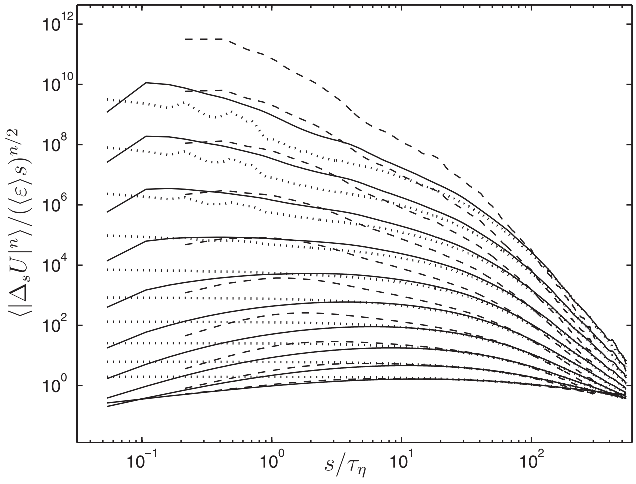

Figure 2. Lagrangian velocity structure function of order n (with n increasing from 1 to 10, bottom to top) for the CCG (solid) and velocity-dissipation (dotted) models compared to data at $R_\lambda \approx 391$ from 1024^3 DNS (dashed).

4. Comparison of CCG model with DNS data

A comparison of CCG model predictions with DNS data for basic conditional and unconditional velocity and acceleration statistics has been reported in [1]. We comment below on the inertial-range behaviour of the CCG model. To this end, we show Lagrangian velocity increment PDFs for different values of the time lag (figure 1) and the associated structure functions (figures 2 and 3). With the choice of equations (12) and (13) for the diffusion coefficient, the Lagrangian velocity increments PDFs are approximately Gaussian at large times but develop stretched tails as the time lag decreases and ultimately approach the Lagrangian acceleration PDF for very small time lags. This behaviour is consistent with recent observations of Lagrangian intermittency in experiments and simulations [7, 8].

We compare CCG model predictions with the velocity-dissipation model (equations (7) and (8)) and with the Langevin equation model. This last model can be obtained after adiabatic elimination of acceleration from the Sawford (1991) model [9] with the result

$$dU = -\frac{U}{T_L^\infty}\,dt + \sqrt{\frac{2\sigma_U^2}{T_L^\infty}}\,dW, \tag{14}$$

where $T_L^\infty = 2\sigma_U^2/(\mathcal{C}_0\langle\varepsilon\rangle)$ ($\mathcal{C}_0$ being the Kolmogorov constant for the second-order Lagrangian velocity structure function). Lagrangian velocity structure functions (from order 1 to 10) for the CCG and velocity-dissipation models are compared to DNS data in figures 2 and 3. As can be seen (figure 2), although the CCG and velocity-dissipation models have been matched with the DNS data for second-order Lagrangian velocity structure functions at large times, the higher order structure functions, too, exhibit good matching with DNS at large times. The velocity-dissipation model predictions in the figure asymptote to constant values at very small times, i.e., the velocity-dissipation model cannot reproduce any intermittency corrections to the Kolmogorov (1941) inertial-range scaling prediction for Lagrangian velocity structure functions. Nevertheless, one finds

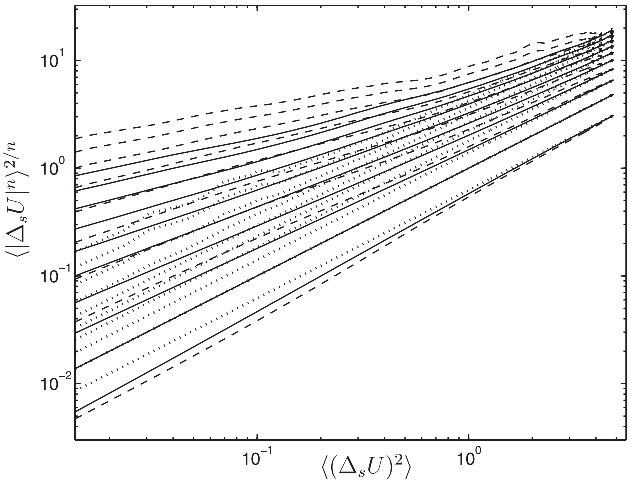

Figure 3. Lagrangian velocity structure function of order n (with n increasing from 1 to 10, bottom to top) for the CCG (solid) and velocity-dissipation (dotted) models compared to data at $R_\lambda \approx 391$ from 1024^3 DNS (dashed).

$$\frac{\langle|\Delta_s U|^n\rangle^{2/n}}{\langle\varepsilon\rangle s} = \frac{4}{3}\left\langle\left[\alpha + \beta\left(\frac{\varphi}{\langle\varepsilon\rangle}\right)^{1/2}\right]^{n/2}\right\rangle^{2/n}, \tag{15}$$

with the velocity-dissipation model as opposed to a constant value for the right-hand side as determined by the Langevin equation. Figure 3 shows the quantity $\langle|\Delta_s U|^n\rangle^{2/n}$ as a function of the second-order Lagrangian velocity structure function for the CCG and velocity-dissipation models compared to DNS. This figure confirms that the velocity-dissipation model provides no accurate description of the DNS data on small-scale two-time statistics. It is notable, however, that for large times and low values of n, the velocity-dissipation model predictions can approximately describe the observations from DNS.

5. Conclusions

After a brief review of the basis for the CCG model, its large-time behaviour is further investigated by means of different assumptions for the multiple-scales procedure which leads to the adiabatic elimination result, equation (11). This relation was derived assuming frozen ϵ-dependences for the coefficients in equation (10). However, additional multiple-scales calculations are presented which support the conclusion that equation (11) has more general validity than suggested by the assumptions made in its derivation.

We briefly examine the intermittent behaviour of the CCG model by showing Lagrangian velocity increments PDFs in the inertial range. Associated Lagrangian velocity structure functions are compared to predictions with the velocity-dissipation model and with the DNS data. This comparison is consistent with previous observations of Lagrangian intermittency in experiments and simulations.

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Appendix A. Regular expansion

First, we show (below) that a regular perturbative treatment of the Fokker–Planck equation (10) leads to a secular divergence at large times, which can be prevented using the method of multiple scales.

Let us now assume that $\mathcal{S}$, $\mathcal{B}$ and h are effectively independent of ϵ so that equation (10) can be rewritten in the form

$$\epsilon\left\{\frac{\partial}{\partial\tau}+\mathscr{S}\right\}f=\mathcal{B}\mathscr{L}_{\bar{a}}f+\frac{2}{3}\epsilon^2 h\mathscr{L}_{\bar{x}}f, \tag{A.1}$$

where $\mathscr{S}\equiv\mathcal{S}[(c_1\bar{a}+c_2\bar{a}^3)\frac{\partial}{\partial\bar{v}}-2c_2\bar{v}\bar{a}-\bar{v}(c_3+c_2\bar{a}^2)\frac{\partial}{\partial\bar{a}}]$. It is not difficult to see that a regular perturbative treatment of (A.1) fails in the large-time limit, i.e.

$$f=f^{(0)}+\epsilon f^{(1)}+\epsilon^2 f^{(2)}+\cdots \tag{A.2}$$

is not uniformly convergent for small ϵ. This can be seen by solving the sequence of problems obtained by substituting (A.2) into (A.1) and equating coefficients of like powers in ϵ. At $O(1)$ equation (A.1) is $\mathscr{L}_{\bar{a}}f^{(0)}=0$, whence

$$f^{(0)}(\bar{v},\bar{x};\tau)=\Phi(\bar{v},\bar{x};\tau)\frac{\mathrm{e}^{-\bar{a}^2/2}}{\sqrt{2\pi}}, \tag{A.3}$$

where Φ is the joint PDF of $(\frac{U}{\sigma_U},\frac{\chi^*}{\sigma_\chi})$ at $O(1)$.† At $O(\epsilon)$ equation (A.1) is

$$\mathcal{B}\mathscr{L}_{\bar{a}}f^{(1)}=\left\{\frac{\partial}{\partial\tau}+\mathscr{S}\right\}f^{(0)}, \tag{A.4}$$

which has to be solved for $f^{(1)}$. For this equation to have a solution, the right-hand side has to be orthogonal to the null space of the adjoint operator $\mathscr{L}^*_{\bar{a}}$ (by the Fredholm alternative theorem). The adjoint operator is $\mathscr{L}^*_{\bar{a}}=(-\bar{a}+\frac{\partial}{\partial\bar{a}})\frac{\partial}{\partial\bar{a}}$ and its null space is spanned by the functions 1 and $\mathrm{i}\,\mathrm{erf}(\frac{\mathrm{i}\bar{a}}{\sqrt{2}})$ (the latter function is unbounded and has to be discarded). Hence, requiring that the integral of the right-hand side of (A.4) with respect to $\bar{a}$ be zero ensures solvability for (A.4). The solvability condition at $O(\epsilon)$ is $\frac{\partial\Phi}{\partial\tau}=0$ and one finds

$$f^{(1)}=-\left(c_3\bar{a}+\frac{c_2}{3}\bar{a}^3\right)\frac{\mathcal{S}}{\mathcal{B}}\frac{\mathrm{e}^{-\bar{a}^2/2}}{\sqrt{2\pi}}\left[\frac{\partial\Phi}{\partial\bar{v}}+\bar{v}\Phi\right]+\Psi(\bar{v},\bar{x};\tau)\frac{\mathrm{e}^{-\bar{a}^2/2}}{\sqrt{2\pi}}, \tag{A.5}$$

where Ψ is the joint PDF of $(\frac{U}{\sigma_U},\frac{\chi^*}{\sigma_\chi})$ at first order. At $O(\epsilon^2)$ equation (A.1) is

$$\mathcal{B}\mathscr{L}_{\bar{a}}f^{(2)}=\left\{\frac{\partial}{\partial\tau}+\mathscr{S}\right\}f^{(1)}-\tfrac{2}{3}h\mathscr{L}_{\bar{x}}f^{(0)}, \tag{A.6}$$

which implies the following solvability condition:

$$\frac{\partial\Psi}{\partial\tau}=\frac{\delta\mathcal{S}^2}{\mathcal{B}}\mathscr{L}_{\bar{v}}\Phi+\tfrac{2}{3}h\mathscr{L}_{\bar{x}}\Phi, \tag{A.7}$$

where $\delta\equiv c_1^2+6c_1c_2+11c_2^2$. However, the solvability condition at $O(\epsilon)$ and (A.7) imply that $\Psi\sim\tau$. Thus, a regular perturbative treatment for (A.1) leads to a secular divergence in the large-time limit. The method of multiple scales must then be used to prevent loss of asymptoticness for the perturbation series (A.2) for $\tau\geqslant O(\epsilon^{-1})$.

Appendix B. Multiple-scales treatment with frozen coefficients

Using the method of multiple scales, the joint PDF in equation (A.1) is treated as a function of several timescales

$$\tau_0=\tau,\quad \tau_1=\epsilon\tau,\quad \tau_2=\epsilon^2\tau,\ldots, \tag{B.1}$$

so that

$$\frac{\partial}{\partial\tau}\to\frac{\partial}{\partial\tau_0}+\epsilon\frac{\partial}{\partial\tau_1}+\epsilon^2\frac{\partial}{\partial\tau_2}+\cdots. \tag{B.2}$$

The function $f(\bar{v},\bar{a},\bar{x};\tau_0,\tau_1,\tau_2,\ldots)$ is then expanded in powers of ϵ as in (A.2) and the solution eventually obtained by restricting the auxiliary time variables to the line (B.1). The fact that $f(\bar{v},\bar{a},\bar{x};\tau_0,\tau_1,\tau_2,\ldots)$ has no physical meaning outside the line (B.1) is exploited to enforce conditions that eliminate the secular divergence at large times.

Substituting (A.2) into the governing equation (A.1) and comparing coefficients of equal powers of ϵ yields a sequence of problems. At $O(1)$ equation (A.1) is

$$\mathscr{L}_{\bar{a}}f^{(0)}=0, \tag{B.3}$$

whence

$$f^{(0)}(\bar{v},\bar{a},\bar{x};\tau_0,\tau_1,\tau_2,\ldots)=\Phi(\bar{v},\bar{x};\tau_0,\tau_1,\tau_2,\ldots)\frac{\mathrm{e}^{-\bar{a}^2/2}}{\sqrt{2\pi}}, \tag{B.4}$$

† It should be noted that since $\mathscr{L}_{\bar{a}}$ is a second-order differential operator, we would expect there to be two linearly independent solutions. A second linearly independent solution is of the form $f^{(0)}=\Psi\frac{\mathrm{e}^{-\bar{a}^2/2}}{\sqrt{2\pi}}\mathrm{i}\,\mathrm{erf}\left(\frac{\mathrm{i}\bar{a}}{\sqrt{2\pi}}\right)$. This is an odd function that does not stay positive and therefore cannot be used as a PDF.

where Φ (with the timescales restricted to (B.1)) is the joint PDF of $(\frac{U}{\sigma_U}, \frac{\chi^*}{\sigma_\chi})$ at leading order. At $O(\epsilon)$ equation (A.1) is

$$\mathcal{B}\mathscr{L}_{\bar{a}} f^{(1)} = \left\{ \frac{\partial}{\partial \tau_0} + \mathscr{S} \right\} f^{(0)}. \tag{B.5}$$

The solvability condition at $O(\epsilon)$ is $\frac{\partial \Phi}{\partial \tau_0} = 0$ and one then finds

$$\begin{aligned} f^{(1)}(\bar{v}, \bar{a}, \bar{x}; \tau_0, \tau_1, \tau_2, \ldots) &= -\left(c_3 \bar{a} + \frac{c_2}{3}\bar{a}^3\right) \frac{\mathcal{S}}{\mathcal{B}} \frac{\mathrm{e}^{-\bar{a}^2/2}}{\sqrt{2\pi}} \left[\frac{\partial \Phi}{\partial \bar{v}} + \bar{v}\Phi\right] \\ &\quad + \Psi(\bar{v}, \bar{x}; \tau_0, \tau_1, \tau_2, \ldots) \frac{\mathrm{e}^{-\bar{a}^2/2}}{\sqrt{2\pi}}, \end{aligned} \tag{B.6}$$

where Ψ (with the timescales restricted to (B.1)) is the joint PDF of $(\frac{U}{\sigma_U}, \frac{\chi^*}{\sigma_\chi})$ at first order. At $O(\epsilon^2)$ equation (A.1) is

$$\mathcal{B}\mathscr{L}_{\bar{a}} f^{(2)} = \left\{ \frac{\partial}{\partial \tau_0} + \mathscr{S} \right\} f^{(1)} + \frac{\partial}{\partial \tau_1} f^{(0)} - \tfrac{2}{3} h \mathscr{L}_{\bar{x}} f^{(0)}, \tag{B.7}$$

which implies the following solvability condition

$$\frac{\partial \Psi}{\partial \tau_0} = -\frac{\partial \Phi}{\partial \tau_1} + \frac{\delta \mathcal{S}^2}{\mathcal{B}} \mathscr{L}_{\bar{v}} \Phi + \tfrac{2}{3} h \mathscr{L}_{\bar{x}} \Phi. \tag{B.8}$$

The right-hand side of this equation does not depend on τ_0 and hence one must impose the condition $\frac{\partial \Psi}{\partial \tau_0} = 0$ to prevent a secular divergence at large τ_0. Combining the solutions to (B.3) and (B.5), the joint PDF of $(\frac{U}{\sigma_U}, \frac{\chi^*}{\sigma_\chi})$ follows as $\rho = \int f \, \mathrm{d}\bar{a}$. Then, the governing equation for ρ is

$$\frac{\partial \rho}{\partial \tau} = \epsilon \left[\frac{\delta \mathcal{S}^2}{\mathcal{B}} \mathscr{L}_{\bar{v}} \rho + \frac{2}{3} h \mathscr{L}_{\bar{x}} \rho \right] + O(\epsilon^2). \tag{B.9}$$

In terms of dimensional variables (and to leading order in ϵ), equation (B.9) can be recast in the form

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \frac{3\delta \mathcal{S}^2}{2\mathcal{B}} \left[\frac{\partial}{\partial v}\left(\frac{v}{T}\rho\right) + \frac{\sigma_U^2}{T} \frac{\partial^2 \rho}{\partial v^2} \right] \\ &\quad + \frac{\partial}{\partial x^*}\left(\frac{x^*}{T_\chi}\rho\right) + \frac{\sigma_\chi^2}{T_\chi} \frac{\partial^2 \rho}{\partial x^{*2}}, \end{aligned} \tag{B.10}$$

which is the Fokker–Planck equation corresponding to the velocity-dissipation model

$$\mathrm{d}U = -\frac{U}{\mathcal{T}_{L,\varphi}} \mathrm{d}t + \sqrt{\frac{2\sigma_U^2}{\mathcal{T}_{L,\varphi}}} \, \mathrm{d}W, \tag{B.11}$$

$$\mathrm{d}\chi^* = -\chi^* \frac{\mathrm{d}t}{T_\chi} + \sqrt{\frac{2\sigma_\chi^2}{T_\chi}} \, \mathrm{d}W', \tag{B.12}$$

with

$$\frac{\mathcal{T}_{L,\varphi}}{T} = \frac{2\mathcal{B}}{3\delta \mathcal{S}^2}. \tag{B.13}$$

This relation shows that $\tau_\eta \bar{b}^2 = 2\mathcal{B}$ can be usefully expressed as the ratio on the right-hand side of (B.13) because that is the velocity timescale of a velocity-dissipation model that is obtained from the CCG model upon formal removal of dissipation-range information (via adiabatic elimination of acceleration).

Appendix C. Multiple-scales procedure with ϵ-dependences in $\mathcal{S}$ and $\mathcal{B}$

In this section, the multiple-scales treatment is modified to account for the ϵ-dependence in $\mathcal{S}$,

$$\mathcal{S} = \sqrt{C_1 \mathcal{S}_1 \epsilon^{0.4} + \ln\left(\frac{C_2}{\epsilon^2}\right) \mathcal{S}_2}, \tag{C.1}$$

with $C_1 = 1.2 \times 15^{-0.1}$, $C_2 = \sqrt{15}/20$, $\mathcal{S}_1 = (\varphi/\langle\varphi\rangle)^{0.15}$ and $\mathcal{S}_2 = (\varphi/\langle\varphi\rangle)^{1.25}$ (equation (C.1) is identical to (1)). The Reynolds-number dependence for $\mathcal{B}$ is unknown and therefore we consider hypothetical dependences that are consistent with the minimal requirement that $\mathcal{B}$ be a decreasing function of Reynolds number (in agreement with the observed increase in intermittency of velocity for increasing Reynolds number) and that make the multiple-scales treatment tractable.

We assume

$$\mathcal{B} = \frac{1}{\sqrt{\ln \epsilon^{-1}}} B, \tag{C.2}$$

where $\mathcal{B} = \frac{1}{2}\tau_\eta \bar{b}^2$, and use a two-term expansion for

$$\mathcal{S} \approx \sqrt{\ln \epsilon^{-1}} \left[1 - \frac{\ln C_2}{4 \ln \epsilon} \right] \underbrace{\sqrt{2\mathcal{S}_2}}_{\mathcal{S}_2^*}, \tag{C.3}$$

valid at asymptotically high Reynolds numbers (i.e. as $\epsilon \to 0$). Thus, equation (10) can be recast in the form:

$$\begin{aligned} &\left\{ \epsilon \sqrt{\ln \epsilon^{-1}} \frac{\partial}{\partial \tau} + \epsilon (\ln \epsilon^{-1}) \mathscr{S}_2 + \epsilon \frac{\ln C_2}{4} \mathscr{S}_2 \right\} f \\ &= B \mathscr{L}_{\bar{a}} f + \frac{2h}{3} \epsilon^2 \sqrt{\ln \epsilon^{-1}} \mathscr{L}_{\bar{x}} f, \end{aligned} \tag{C.4}$$

where $\mathscr{S}_2 = \mathcal{S}_2^* [(c_1 \bar{a} + c_2 \bar{a}^3) \frac{\partial}{\partial \bar{v}} - 2c_2 \bar{v}\bar{a} - \bar{v}(c_3 + c_2 \bar{a}^2) \frac{\partial}{\partial \bar{a}}]$. (Additionally,

$$h(\epsilon) \approx 18.18 - 454.85 \, \epsilon^{1.4} + \cdots, \quad \text{as } \epsilon \to 0; \tag{C.5}$$

however, we make $h \equiv h(\epsilon = 0)$ in (C.4) because we are only interested in the solvability conditions at second order.)

The joint PDF in equation (C.4) is treated as a function of several timescales

$$\tau_0 = \tau, \ldots \qquad \tau_{ij} = \epsilon^i (\ln \epsilon^{-1})^{j/2} \tau, \ldots \tag{C.6}$$

so that

$$\frac{\partial}{\partial \tau} \to \frac{\partial}{\partial \tau_0} + \sum_{i=1}^{\infty} \sum_{j=-\infty}^{\infty} \epsilon^i (\ln \epsilon^{-1})^{j/2} \frac{\partial}{\partial \tau_{ij}}. \tag{C.7}$$

The function $f(\bar{v}, \bar{a}, \bar{x}; \tau_0, \ldots, \tau_{ij}, \ldots)$ is then expanded in powers of ϵ and $\ln \epsilon^{-1}$,

$$f = f^{(0)} + \sum_{i=1}^{\infty} \sum_{j=-\infty}^{\infty} \epsilon^i (\ln \epsilon^{-1})^{j/2} f^{(i,j)} \tag{C.8}$$

and the solution eventually obtained by restricting the auxiliary time variables to the line (C.6). Substituting (C.8) into the governing equation (C.4) and comparing coefficients of equal powers of ϵ and $\ln \epsilon^{-1}$ yields a sequence of problems. At $O(1)$ equation (C.4) is

$$\mathscr{L}_{\bar{a}} f^{(0)} = 0, \tag{C.9}$$

whence

$$f^{(0)}(\bar{v},\bar{a},\bar{x};\tau_0,\tau_1,\tau_2,\ldots)=\Phi(\bar{v},\bar{x};\tau_0,\tau_1,\tau_2,\ldots)\frac{\mathrm{e}^{-\bar{a}^2/2}}{\sqrt{2\pi}}, \tag{C.10}$$

where Φ (with the timescales restricted to (C.6)) is the joint PDF of $(\frac{U}{\sigma_U},\frac{\chi^*}{\sigma_\chi})$ at $O(1)$. At $O(\epsilon(\ln\epsilon^{-1})^{j/2})$ equation (C.4) is

$$B\mathscr{L}_{\bar{a}}f^{(1,j)}=\frac{\partial f^{(0)}}{\partial\tau_0}\delta_{j,1}+\mathscr{S}_2 f^{(0)}\delta_{j,2}+\frac{\ln C_2}{4}\mathscr{S}_2 f^{(0)}\delta_{j,0}. \tag{C.11}$$

The only solvability condition at $O(\epsilon)$ is $\frac{\partial\Phi}{\partial\tau_0}=0$ and one then finds

$$f^{(1,j)}=-\left[\left(\frac{\ln C_2}{4}-1\right)\delta_{j,0}+1\right]\left[c_3\bar{a}+\frac{c_2}{3}\bar{a}^3\right]\frac{\mathcal{S}_2^*}{B}\frac{\mathrm{e}^{-\bar{a}^2/2}}{\sqrt{2\pi}}\times\left[\frac{\partial\Phi}{\partial\bar{v}}+\bar{v}\Phi\right](\delta_{j,0}+\delta_{j,2})+\Phi^{(1,j)}\frac{\mathrm{e}^{-\bar{a}^2/2}}{\sqrt{2\pi}}, \tag{C.12}$$

where $\Phi^{(1,j)}$ (with the timescales restricted to (C.6)) is the joint PDF of $(\frac{U}{\sigma_U},\frac{\chi^*}{\sigma_\chi})$ at first order. At $O(\epsilon^2(\ln\epsilon^{-1})^{j/2})$ equation (C.4) is

$$B\mathscr{L}_{\bar{a}}f^{(2,j)}=\frac{\partial f^{(1,j-1)}}{\partial\tau_0}+\frac{\partial f^{(0)}}{\partial\tau_{1,j-1}}+\mathscr{S}_2 f^{(1,j-2)}+\frac{\ln C_2}{4}\mathscr{S}_2 f^{(1,j)}-\tfrac{2}{3}h\mathscr{L}_{\bar{x}}f^{(0)}\delta_{j,1}, \tag{C.13}$$

which implies the following solvability conditions:

$$\frac{\partial\Phi^{(1,j)}}{\partial\tau_0}+\frac{\partial\Phi}{\partial\tau_{1,j}}=\left[\delta_{j,3}+\delta_{j,1}\frac{\ln C_2}{2}+\delta_{j,-1}\left(\frac{\ln C_2}{4}\right)^2\right]\times\frac{\delta\mathcal{S}_2^{*2}}{B}\mathscr{L}_{\bar{v}}\Phi+\tfrac{2}{3}h\delta_{j,0}\mathscr{L}_{\bar{x}}\Phi, \tag{C.14}$$

where $\delta\equiv c_1^2+6c_1c_2+11c_2^2$. Combining the solutions to (C.9) and (C.11), the joint PDF of $(\frac{U}{\sigma_U},\frac{\chi^*}{\sigma_\chi})$ follows as $\rho=\int f\,\mathrm{d}\bar{a}$. In terms of dimensional variables (and to leading order in ϵ), the governing equation for ρ is

$$\frac{\partial\rho}{\partial t}=\sqrt{\ln\epsilon^{-1}}\frac{3\delta}{2B}\left[\ln\left(\frac{C_2}{\epsilon^2}\right)\right]\mathcal{S}_2\left[\frac{\partial}{\partial v}\left(\frac{v}{T}\rho\right)+\frac{\sigma_U^2}{T}\frac{\partial^2\rho}{\partial v^2}\right]+\frac{\partial}{\partial x^*}\left(\frac{x^*}{T_\chi}\rho\right)+\frac{\sigma_\chi^2}{T_\chi}\frac{\partial^2\rho}{\partial x^{*2}}. \tag{C.15}$$

This is the Fokker–Planck equation associated with equations (B.11) and (B.12) with

$$\frac{\mathcal{T}_{L,\varphi}}{T}=\frac{2B/\sqrt{\ln\epsilon^{-1}}}{3\delta\ln\left(\frac{C_2}{\epsilon^2}\right)\mathcal{S}_2}, \tag{C.16}$$

or

$$\frac{\mathcal{T}_{L,\varphi}}{T}=\frac{2\mathcal{B}}{3\delta\ln\left(\frac{R_\lambda}{20}\right)\mathcal{S}_2}. \tag{C.17}$$

This expression is identical to that given by equation (B.13) when only the second term on the right-hand side of (C.1) is considered.

In another multiple-scales treatment for the CCG model based on

$$\mathcal{B}=\sqrt{\epsilon\ln\epsilon^{-1}}B \tag{C.18}$$

(in place of (C.2)) and (C.3), the same result (i.e. equation (C.16)) is obtained.

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