STOCHASTIC MODEL FOR TURBULENT FREQUENCY [1]

by

Jayesh and S.B. Pope

FDA 95-05 October 1995

Fluid Dynamics and Aerodynamics Program

Sibley School of
Mechanical and Aerospace Engineering

Cornell University Ithaca, New York 14853

ABSTRACT

A simple stochastic model is described for the turbulent frequency following a fluid particle in turbulent flow. This model is simpler than the log-normal model of Pope and Chen (1990), and has advantages in regions of turbulent/non-turbulent intermittent flow. In stationary isotropic turbulence, the model results in the pdf of frequency being a Gamma distribution, with specified normalized variance; and the autocorrelation is an exponential with specified timescale. The model is intended for use in PDF methods based on the joint pdf of velocity, frequency and other fluid variables.

INTRODUCTION

Pope and Chen (1990) and Pope (1991) describe a stochastic model for the turbulence frequency $\omega(t)$ following a fluid particle. Though physically accurate (compared to DNS data), the model has proved to have several undesirable features when used in PDF methods. Specifically:

1. the model is complicated (Pope 1995) and computationally expensive to implement

2. the underlying pdf is the log-normal distribution, whose very long tails lead to substantial statistical fluctuations in Monte Carlo implementations

3. an *ad hoc* additional term is needed to effect entrainment of non-turbulent fluid.

The model developed here overcomes these problems.

For inhomogeneous flows, in addition to the mean frequency $\langle\omega\rangle$, a conditional mean frequency Ω is defined. In the intermittent region of free shear flows, Ω remains appreciable as $\langle\omega\rangle$ tends to zero. The use of Ω (rather than $\langle\omega\rangle$) as the effective rate of turbulent processes is found to improve the model performance.

In the next section the model is described for stationary turbulence. The conditional mean Ω is then defined, and the general form of the model is given.

This model is incorporated in the code PDF2DV (Pope 1994). It appears to be extremely robust, and performs (qualitatively) satisfactorily on all tests performed to date. An illustrative computation is reported. Quantitative results will be described in subsequent papers.

STATIONARY PROCESS

The starting point for the description of the model is statistically-stationary, homogeneous turbulence in which $\omega(t)$ is a stationary process. As in Pope and Chen (1990), $\omega(t)$ is modelled as a diffusion process

$$d\omega = A(\omega)\, dt + \sqrt{B(\omega)}\, dW, \tag{1}$$

where $A(\omega)$ and $B(\omega)$ are the drift and diffusion coefficients, and $W(t)$ is a Wiener process. The specific forms chosen for $A(\omega)$ and $B(\omega)$—for reason to be explained—are

$$A(\omega) = -(\omega - \langle\omega\rangle)/T, \tag{2}$$

and

$$B(\omega) = 2\sigma^2 \langle\omega\rangle \omega / T, \tag{3}$$

where T is a specified time scale, and σ^2 is the normalized variance

$$\sigma^2 = \mathrm{var}(\omega)/\langle\omega\rangle^2. \tag{4}$$

Straightforward analysis shows that, according to these equations, $\langle\omega\rangle$ and $\langle\omega^2\rangle$ are indeed stationary, that the variance is given by Eq. (4), and that the autocovariance is

$$\langle(\omega(t) - \langle\omega\rangle)(\omega(t+s) - \langle\omega\rangle)\rangle = \exp(-|s|/T). \tag{5}$$

To appreciate the selection of $A(\omega)$ and $B(\omega)$, we consider the pdf of $\omega, f(\eta)$, where η is the sample-space variable. The Fokker-Planck equation (Gardiner 1985) corresponding to Eq. (1) is (for this stationary process)

$$0 = -\frac{d}{d\eta}[f(\eta)A(\eta)] + \frac{1}{2}\frac{d^2}{d\eta^2}[f(\eta)B(\eta)]. \tag{6}$$

Integrating twice, we obtain the pdf

$$f(\eta) = \frac{C}{B} \exp\left\{\int \frac{2A\,d\eta}{B}\right\}, \tag{7}$$

where C is a constant determined by the normalization condition

$$\int_{-\infty}^{\infty} f(\eta)\,d\eta = 1. \tag{8}$$

For simplicity, for ease of analysis and implementation, and to facilitate variance reduction techniques, $A(\omega)$ is taken to be linear in ω, i.e., Eq. (2). This leaves B to determine the shape of the pdf. The choice of $B =$ constant leads to the Langevin equation, and f being Gaussian.

Since ω is non-negative, we want $f(\eta)$ to increase from zero at $\eta = 0$, and for $\omega = 0$ to be an "unattainable entry boundary" (Karlin and Taylor 1981). This is achieved with Eq. (3). With $A(\omega)$ and $B(\omega)$ given by Eq. (2) and Eq. (3), the stationary pdf is

$$f(\eta) = C\eta^{\frac{1}{\sigma^2}-1} \exp\left\{\frac{-\eta}{\sigma^2\langle\omega\rangle}\right\}. \tag{9}$$

For $0 < \sigma^2 < 1, f(0) = 0$, and f decay exponentially for large η.

It may be recognized that Eq. (9) is a Gamma distribution, which is put in standard form by defining

$$X = \frac{\omega}{\sigma^2\langle\omega\rangle}, \tag{10}$$

and

$$p = \frac{1}{\sigma^2}. \tag{11}$$

Then the pdf of $X, g(x)$, is

$$g(x) = \frac{1}{\Gamma(p)} x^{p-1} e^{-x}. \tag{12}$$

Note that $\langle X\rangle = \text{var}(X) = p$. By inverting this transformation, the normalization constant is identified, so that Eq. (9) can be rewritten

$$f(\eta) = \left\{\Gamma\left(\frac{1}{\sigma^2}\right) \left(\sigma^2\langle\omega\rangle\right)^{\frac{1}{\sigma^2}}\right\}^{-1} \eta^{\frac{1}{\sigma^2}-1} \exp\left(\frac{-\eta}{\sigma^2\langle\omega\rangle}\right). \tag{13}$$

To yield approximate correspondence with the DNS data of Yeung and Pope (1989), we take

$$\sigma^2 = \frac{1}{4}, \tag{14}$$

and

$$T^{-1} = C_3 \langle \omega \rangle, \tag{15}$$

with

$$C_3 = 1.0. \tag{16}$$

Figure 1 shows $f(\eta)$ according to Eqs. (13) and (14).

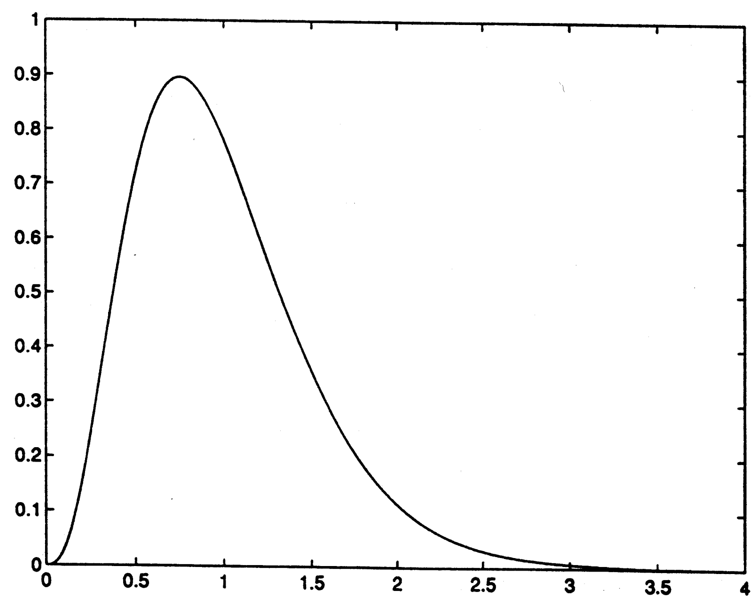

Figure 1: The pdf of ω for stationary turbulence (Eq. 13, with $\langle \omega \rangle = 1, \sigma^2 = \frac{1}{4}$).

CONDITIONAL MEAN FREQUENCY

In an intermittent region of a turbulent flow, with probability γ the fluid is in turbulent motion, and with probability $(1-\gamma)$ it is in non-turbulent irrotational motion: γ is the intermittency factor. In non-turbulent fluid, ω is zero. Hence, in the intermittent region, the pdf of ω, $f(\eta)$, can be written

$$f(\eta) = (1-\gamma)\delta(\eta) + \gamma f_T(\eta), \tag{17}$$

where $f_T(\eta)$ is the pdf of ω in the turbulent fluid.

Arguably, the appropriate rate to model processess in the turbulent fluid is the conditional, turbulent mean

$$\begin{aligned} \langle\omega\rangle_T &\equiv \int_0^\infty \eta f_T(\eta)\, d\eta \\ &= \langle\omega\rangle/\gamma. \end{aligned} \tag{18}$$

Clearly, towards the non-turbulent edge of the intermittent region ($\gamma < 0\cdot 1$, say) $\langle\omega\rangle_T$ is much larger than $\langle\omega\rangle$, which is the rate usually used in models.

Both experimentally and in modelling it is problematic to distinguish unambiguously between turbulent and non-turbulent fluid—a threshold is required. As a well-conditioned surrogate for $\langle\omega\rangle_T$, we use the conditional—or *above-average mean*—

$$\begin{aligned} \Omega &\equiv C_\Omega \langle \omega | \omega \geq \langle\omega\rangle\rangle \\ &= C_\Omega \int_{\langle\omega\rangle}^\infty \eta f(\eta)\, d\eta \Bigg/ \int_{\langle\omega\rangle}^\infty f(\eta)\, d\eta\,. \end{aligned} \tag{19}$$

The constant C_Ω is chosen so that Ω equals $\langle\omega\rangle$ when the pdf $f(\eta)$ is given by Eq. (13). This consideration yields

$$C_\Omega = Q(p,p)/Q(p+1,p), \tag{20}$$

where $p = \sigma^{-2}$, and Q is the incomplete gamma function defined in Press, Teukolsky, Vetterling, and Flannery (1992),

$$Q(a,x) = \Gamma(a,x)/\Gamma(a). \tag{21}$$

For $\sigma^2 = \frac{1}{4}$, the value of C_Ω is

$$C_\Omega \approx 0.6893. \tag{22}$$

GENERAL MODEL

The general model for inhomogeneous flows differs from the stationary model in just two respects. First, the definition of T (Eq. 19) is replaced by

$$T^{-1} = C_3 \Omega, \tag{23}$$

and, second, a source term—related to that in the dissipation equation—is added. The result is

$$\begin{aligned} d\omega &= -(\omega - \langle\omega\rangle)\frac{dt}{T} - \langle\omega\rangle\omega S_\omega \, dt \\ &\quad +\{2\sigma^2\langle\omega\rangle\omega/T\}^{\frac{1}{2}} \, dW. \end{aligned} \tag{24}$$

The non-dimensional source term S_ω is as defined by Pope (1991):

$$S_\omega = C_2 - C_1 S_{ij} S_{ij} / \langle\omega\rangle^2, \tag{25}$$

where S_{ij} is the mean rate of strain

$$S_{ij} \equiv \frac{1}{2}\left(\frac{\partial\langle U_i\rangle}{\partial x_j} + \frac{\partial\langle U_j\rangle}{\partial x_i}\right), \tag{26}$$

and the constants are taken to be $C_2 = 0.9$ and $C_1 = 0.08$.

RESULTS

As an example of the use of the model, Figure 2 shows calculations of the temporal mixing layer performed by Delarue (personal communication) using the code PDF2DV (Pope 1994). Notice in particular the skirt on the profile of Ω.

ACKNOWLEDGMENTS

This work was supported in part by the Air Force Office of Scientific Research, Grant F49620-94-1-0098 and in part by contract F3315-87-2821 from the Air Force Wright Laboratories to the Allison Engine Company. We are grateful to Bertrand Delarue for permission to use his unpublished results.

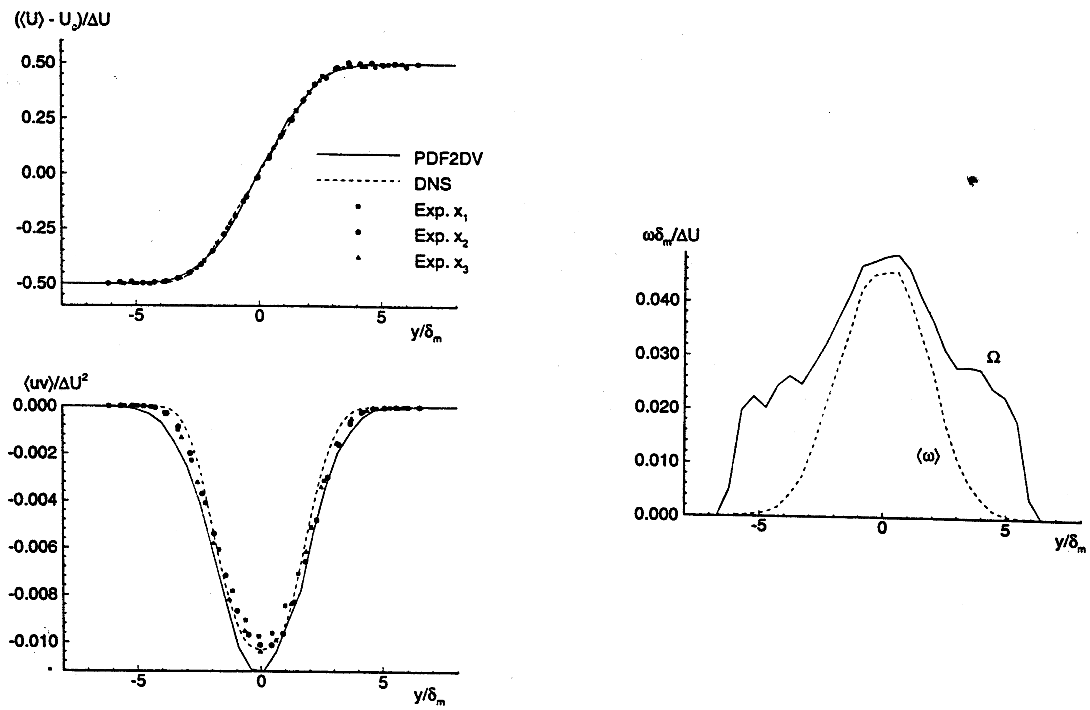

Figure 2: Temporal mixing layer showing (a) mean velocity profile (b) shear stress profile (c) profiles of mean $\langle\omega\rangle$ and conditional mean Ω frequency. Symbols, experimental data of Bell and Mehta (1990); dashed line, DNS of Rogers and Moser (1994); solid line, PDF model calculations using the stochastic frequency model (Delarue, private communication).

References

Bell, J. H. and R. D. Mehta (1990). Developmant of a two-stream mixing layer from tripped and untripped boundary layers. *AIAA J. 28*, 2034–2042.

Gardiner, C. W. (1985). *Handbook of stochastic methods for physics, chemistry and the natural sciences* (Second ed.). Berlin: Springer-Verlag.

Karlin, S. and H. M. Taylor (1981). *A second course in stochastic processes.* New York: Academic Press.

Pope, S. B. (1991). Application of the velocity-dissipation PDF model to inhomogeneous turbulent flows. *Phys. Fluids A 3*, 1947–1957.

Pope, S. B. (1994). PDF2DV: A Fortran code to solve the modelled joint PDF equations for two-dimensional recirculating flows. Unpublished, Cornell University.

Pope, S. B. (1995). Particle method for turbulent flows: Integration of stochastic model equations. *J. Comp. Phys 117*, 332–349.

Pope, S. B. and Y. L. Chen (1990). The velocity-dissipation PDF model for turbulent flows. *Phys. Fluids A 2*, 1437–1449.

Press, W. H., S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery (1992). *Numerical Recipes in Fortran* (Second ed.). Cambridge.

Rogers, M. M. and R. D. Moser (1994). Direct simulation of a self-similar turbulent mixing layer. *Phys. Fluids 6*, 903–923.

Yeung, P. K. and S. B. Pope (1989). Lagrangian statistics from direct numerical simulations of isotropic turbulence. *J. Fluid Mech. 207*, 531–586.