PDF AND REYNOLDS STRESS MODELING OF NEAR-WALL TURBULENT FLOWS

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ABSTRACT

It is well established that pdf methods are especially suited to calculations of turbulent reactive flows because the reaction terms appear exactly in the governing equations (Pope,1985). The ultimate aim of the model developed here is to extend pdf methods so that they can include accurate modeling of near-wall turbulence in such calculations. A Stochastic Lagrangian Wall (SLW) model is developed in which the Generalized Langevin model (Haworth,1986,1987) is combined with an exact model for viscous transport. Then the method of elliptic relaxation (Durbin,1991,1993) is used to incorporate the wall effects without the use of wall functions or damping functions. Information about the proximity of the wall is provided only in the boundary conditions so that the model can be implemented without ad hoc assumptions about the geometry of the flow. A near-wall Reynolds-stress closure is derived from this SLW model, and predictions are compared with DNS results for fully developed channel flow.

NEAR-WALL REYNOLDS STRESSES

It is important for any near-wall model to accurately represent the behavior of the Reynolds-stresses near the wall. Here we use a standard Taylor series analysis (Mansour,1988) to examine the near-wall Reynolds-stress equations and their solutions. Results of this analysis are important in the construction of the SLW model. We split the velocity and pressure into the familiar Reynolds decomposition:

$$U_i = \langle U_i \rangle + u_i, \tag{1}$$
$$\mathcal{P} = \langle \mathcal{P} \rangle + p. \tag{2}$$

For

$$\frac{\tilde{D}(\cdot)}{Dt} = \frac{\partial(\cdot)}{\partial t} + \langle U_k \rangle \frac{\partial(\cdot)}{\partial x_k}, \tag{3}$$

the exact Reynolds-stress equation is

$$\frac{\tilde{D}\langle u_i u_j \rangle}{Dt} = T_{(v)ij} + T_{(t)ij} + P_{ij} + \phi_{ij} - \epsilon_{ij}, \tag{4}$$

where the terms on the right-hand side represent viscous transport, turbulent transport, production, velocity-pressure gradient correlations, and dissipation, respectively. It is important to note that ϕ_{ij} includes both pressure transport and redistribution. Though the analysis is valid for any wall bounded flow, coordinates are arranged here to accommodate one dimensional flow: x_1 is aligned with the mean flow, x_2 is wall-normal, and x_3 is in the spanwise direction. To describe the near-wall behavior, the dependent variables are expanded in a Taylor series about the wall (Mansour,1988). Let $y = x_2$ be the distance from the wall, and let $a_i, b_i, c_i, \ldots, i = 1,2,3$ and $a_p, b_p, c_p, \ldots$ be random functions of x_1, x_3, and time. Then as long as u_i and p are analytic functions of y, the fluctuating velocities are

$$u_1 = a_1 + b_1 y + c_1 y^2 + \ldots \tag{5}$$
$$u_2 = a_2 + b_2 y + c_2 y^2 + \ldots \tag{6}$$
$$u_3 = a_3 + b_3 y + c_3 y^2 + \ldots \tag{7}$$
$$p = a_p + b_p y + c_p y^2 + \ldots \tag{8}$$

If we impose the boundary conditions of no slip and impermeability, and the governing equations for conservation of mass and momentum, we find the following results for the Reynolds-stresses: First, to leading order, the scaling of each Reynolds-stress with distance from the wall is

$$\langle u_1^2 \rangle \sim y^2, \tag{9}$$

i,j	$T_{(v)ij}$	ϕ_{ij}	ϵ_{ij}
1,1	$2\nu \left\langle b_1^2 \right\rangle$	$O(y)$	$2\nu \left\langle b_1^2 \right\rangle$
2,2	$12\nu \left\langle c_2^2 \right\rangle y^2$	$-2 \langle c_2 b_p \rangle y^2$	$8\nu \left\langle c_2^2 \right\rangle y^2$
3,3	$2\nu \left\langle b_3^2 \right\rangle$	$O(y)$	$2\nu \left\langle b_3^2 \right\rangle$
1,2	$6\nu \langle b_1 c_2 \rangle y$	$-\langle b_1 b_p \rangle y$	$4\nu \langle b_1 c_2 \rangle y$

Table 1: Near-wall terms as a function of y to first order

$$\left\langle u_2^2 \right\rangle \sim y^4, \tag{10}$$
$$\left\langle u_3^2 \right\rangle \sim y^2, \tag{11}$$
$$\langle u_1 u_2 \rangle \sim y^3. \tag{12}$$

Second, we find that the dominant balance of Reynolds-stress terms near the wall becomes

$$T_{(v)ij} + \phi_{ij} - \epsilon_{ij} = 0. \tag{13}$$

The scaling with y of these dominant terms is given for each Reynolds-stress equation in Table 1. And third, Taylor series expansions for $k = \frac{1}{2}\langle u_i u_i \rangle$ and $\epsilon = \frac{1}{2}\epsilon_{ii}$ can be used to express the near-wall Reynolds-stress equations to first order in closed form:

$$\nu \frac{\partial^2 \left\langle u_1^2 \right\rangle}{\partial y^2} - \frac{\epsilon}{k}\left\langle u_1^2 \right\rangle = 0, \tag{14}$$
$$\nu \frac{\partial^2 \left\langle u_2^2 \right\rangle}{\partial y^2} - 6\frac{\epsilon}{k}\left\langle u_2^2 \right\rangle = 0, \tag{15}$$
$$\nu \frac{\partial^2 \left\langle u_3^2 \right\rangle}{\partial y^2} - \frac{\epsilon}{k}\left\langle u_3^2 \right\rangle = 0, \tag{16}$$
$$\nu \frac{\partial^2 \langle u_1 u_2 \rangle}{\partial y^2} - 3\frac{\epsilon}{k}\langle u_1 u_2 \rangle = 0. \tag{17}$$

To first order, these are exact equations for the dominant physical processes which govern the flow near the wall. Viscous transport appears in the first term of each equation. For $\left\langle u_1^2 \right\rangle$ and $\left\langle u_3^2 \right\rangle$, dissipation is represented in the second term of each equation. For $\left\langle u_2^2 \right\rangle$ and $\langle u_1 u_2 \rangle$, the sum of dissipation and the pressure correlations are represented in the second terms. To model the flow accurately, it is important for a near-wall Reynolds-stress closure to come as close as possible to this set of equations as $y \to 0$.

The above results bring out the strong anisotropy and inhomogeneity which need to be addressed in the modeling of near-wall flows. The anisotropy is clear from the terms in Table (1). The dominant processes are of order (1) at the wall in the $(1,1)$ and $(3,3)$ directions, but they vanish in the directions which have a normal component; the turbulence becomes two dimensional as the wall is approached. Summing the diagonal terms of the Reynolds-stress equations in Table (1) shows that the contributions to the dissipation of kinetic energy near the wall are only important in the $(1,1)$ and $(3,3)$ directions. The inhomogeneity in the wall normal direction appears in Eqs. (14)-(17) in which the viscous transport dominates the balance of every Reynolds-stress component. Elliptic relaxation (Durbin,1993) has been shown to model these near-wall effects well in Reynolds-stress closures; we hope to capture these effects by incorporating elliptic relaxation in the pdf approach.

PDF EVOLUTION EQUATION WITH MOLECULAR VISCOSITY

Let $\mathbf{U}(\mathbf{x},t)$ and $\mathcal{P}(\mathbf{x},t)$ be the Eulerian velocity and pressure fields, both governed by the Navier Stokes equations. Now let the Eulerian pdf, $f(\mathbf{V};\mathbf{x},t)$ be the pdf of velocity at a given location. The evolution equation for the Eulerian pdf can be expressed in two ways (Pope,1985):

$$\frac{\partial f}{\partial t} + V_i \frac{\partial f}{\partial x_i} = \frac{1}{\rho}\frac{\partial \langle \mathcal{P} \rangle}{\partial x_i}\frac{\partial f}{\partial V_i} + \frac{\partial}{\partial V_i}\left[f \left\langle \frac{1}{\rho}\frac{\partial p}{\partial x_i} - \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j} | \mathbf{U}(\mathbf{x},t) = \mathbf{V} \right\rangle \right] \tag{18}$$

or

$$\begin{aligned} \frac{\partial f}{\partial t} + V_i \frac{\partial f}{\partial x_i} &= \nu \frac{\partial^2 f}{\partial x_j \partial x_j} + \frac{1}{\rho}\frac{\partial \langle \mathcal{P} \rangle}{\partial x_i}\frac{\partial f}{\partial V_i} \\ &- \frac{\partial^2}{\partial V_i \partial V_j}\left[f \left\langle \nu \frac{\partial U_i}{\partial x_k}\frac{\partial U_j}{\partial x_k} | \mathbf{U}(\mathbf{x},t) = \mathbf{V} \right\rangle \right]. \\ &+ \frac{\partial}{\partial V_i}\left[f \left\langle \frac{1}{\rho}\frac{\partial p}{\partial x_i} | \mathbf{U}(\mathbf{x},t) = \mathbf{V} \right\rangle \right] \end{aligned} \tag{19}$$

For modeling in which viscosity is important, we will use Eq. (19), because the first term on its right hand side represents viscous diffusion exactly. That term leads to the important viscous terms in the near-wall balances of Eqs. (14)-(17). Only the terms on the first line of Eq. (19) are in closed form, so an appropriate pdf model must provide a closed approximation for its remaining terms.

STOCHASTIC LAGRANGIAN WALL MODEL

The SLW model provides both a pdf and a Reynolds-stress closure. Here, we construct both versions in four stages. First, particle equations which represent the Navier-Stokes equations exactly are derived in a way which leads to modeling of viscous transport. Second, the Generalized Langevin model is used to close the unknown terms of the particle equations. The important parameters in this model are specified using Durbin's elliptic relaxation technique. Third, we present the resulting pdf and Reynolds-stress models. The pdf model is completed by closing Eq. (19) with a modeled pdf evolution equation, and by providing an equation for dissipation. We find the Reynolds-stress version by deriving velocity moment equations from the modeled pdf equation, and by adding a model for turbulent transport. Fourth, Reynolds-stress boundary conditions are imposed and the near-wall behavior is compared with the correct near-wall behavior described previously. This model development is guided by two main priorities:

1 - Far from the wall, the SLW model approaches the familiar Isotropization of Production (IP) model (Naot,1970,Launder,1975).

2 - Close to the wall, the dominant balance of terms as represented in Eq. (13) is modeled as accurately as possible.

Exact Particle Equations

For the Eulerian velocity field $\mathbf{U}(\mathbf{x},t)$, consider an ensemble of particles moving through the field with position $\mathcal{X}(t)$ and velocity $\mathcal{U}(t)$. For $\mathbf{U}(\mathbf{x},t)$ governed by the Navier-Stokes equations, we can define the particle velocity as the Eulerian velocity evaluated at the particle position:

$$\mathcal{U}(t) = \mathbf{U}[\mathcal{X}(t),t]\,. \tag{20}$$

We seek equations to govern the position and velocity of these particles in which the viscous stress can be modeled exactly.

Previous pdf formulations have used fluid particles. With this approach, a small change in particle position $d\mathcal{X}$ over a small time interval dt is determined by the local fluid velocity (Pope,1985,Haworth,1986):

$$d\mathcal{X}_i = \mathcal{U}_i dt. \tag{21}$$

By this definition, fluid particles are convected through the velocity field; consequently these models are able to represent convective transport exactly. But fluid particles are inadequate for capturing viscous transport, because any pdf evolution equation which follows from Eq. (21) (such as Eq. (18)) must exclude the important viscous term $\nu\frac{\partial^2 f}{\partial x_i \partial x_i}$, which appears in Eq. (19) (see for example Haworth,1986). In an effort to capture the effects of both convective and viscous transport, we consider stochastic particles which undergo both convective and molecular motion. So we combine Eq. (21) with the classical model for Brownian motion (Einstein,1926) in which the change in position of a molecule is governed by a symmetric probability distribution. To characterize this random motion, we use the isotropic Wiener process $d\mathbf{W}$ in which increments have a normal distribution with zero mean, and

$$\langle dW_i dW_j\rangle = dt\,\delta_{ij}. \tag{22}$$

Then increments of particle position are given by

$$d\mathcal{X}_i = \mathcal{U}_i dt + \sqrt{2\nu}\,dW_i, \tag{23}$$

where $\sqrt{2\nu}$ is chosen to insure that the momentum carried by these particles diffuses in physical space with coefficient ν. This particle motion leads to the viscous transport term in Eq. (19), and hence to the viscous terms in the near-wall balances of Eqs. (14-17).

We now find the corresponding increment of particle velocity $d\mathcal{U}$. For an arbitrary change in position $d\mathcal{X}$ over the small time interval dt, we have

$$d\mathcal{U}_i = \frac{\partial U_i}{\partial t}dt + \frac{\partial U_i}{\partial x_j}d\mathcal{X}_j + \frac{1}{2}\frac{\partial^2 U_i}{\partial x_j \partial x_k}d\mathcal{X}_j d\mathcal{X}_k + \dots \tag{24}$$

We substitute Eq. (23) into Eq. (24) and retain the terms up to order dt:

$$d\mathcal{U}_i = \frac{\partial U_i}{\partial t}dt + \frac{\partial U_i}{\partial x_j}\left(\mathcal{U}_j dt + \sqrt{2\nu}\,dW_j\right) + \nu\frac{\partial^2 U_i}{\partial x_j \partial x_j}dt. \tag{25}$$

Note that the Wiener process in Eq. (25) is identical to the one which appears in the equation for position, Eq. (23). Next we rewrite the first two terms on the right hand side of Eq. (25), using Eq. (20) and the Navier Stokes equations:

$$d\mathcal{U}_i = -\frac{1}{\rho}\frac{\partial \mathcal{P}}{\partial x_i}dt + 2\nu\frac{\partial^2 U_i}{\partial x_j \partial x_j}dt + \sqrt{2\nu}\frac{\partial U_i}{\partial x_j}dW_j. \tag{26}$$

Eq. (26) describes the change in velocity of a particle whose position evolves by Eq. (23) through a velocity field which is governed by the Navier Stokes equations.

Generalized Langevin Model with Elliptic Relaxation

We now construct model particle equations to approximate Eqs. (23) and (26). Throughout this section we denote modeled physical quantities and pdf's with a "*" superscript to distinguish them from their exact counterparts. Let $f^*(\mathbf{V};\mathbf{x},t)$ be the modeled pdf of velocity at a given location $\mathbf{x}$. Then mean velocities are defined as

$$\langle \mathbf{U}_i^*(\mathbf{x},t)\rangle = \int V_i f^* d\mathbf{V}, \tag{27}$$

with the integral taken over all of velocity space. For modeling in a context in which Eulerian velocities are unknown, Eq. (26) is divided into closed and unclosed terms with the Reynolds decomposition:

$$\begin{aligned} d\mathcal{U}_i \quad = \quad & -\frac{1}{\rho}\frac{\partial \langle\mathcal{P}\rangle}{\partial x_i}dt + 2\nu\frac{\partial^2 \langle U_i\rangle}{\partial x_j \partial x_j}dt + \sqrt{2\nu}\frac{\partial \langle U_i\rangle}{\partial x_j}dW_j \\ & -\frac{1}{\rho}\frac{\partial p}{\partial x_i}dt + 2\nu\frac{\partial^2 u_i}{\partial x_j \partial x_j}dt + \sqrt{2\nu}\frac{\partial u_i}{\partial x_j}dW_j. \end{aligned} \tag{28}$$

Then we define the motion of modeled particles with position $\mathcal{X}^*(t)$ and velocity $\mathcal{U}^*(t)$. Increments of position are given by Eq. (23), and increments in velocity are defined by using the Generalized Langevin model to replace the unclosed terms of Eq. (28):

$$d\mathcal{X}_i^* \quad = \quad \mathcal{U}_i^* dt + \sqrt{2\nu}\,dW_i \tag{29}$$

$$\begin{aligned} d\mathcal{U}_i^* \quad = \quad & -\frac{1}{\rho}\frac{\partial \langle\mathcal{P}^*\rangle}{\partial x_i}dt + 2\nu\frac{\partial^2 \langle U_i^*\rangle}{\partial x_j \partial x_j}dt + \sqrt{2\nu}\frac{\partial \langle U_i^*\rangle}{\partial x_j}dW_j \\ & + \quad G_{ij}\left(\mathcal{U}_j^* - \langle U_j^*\rangle\right)dt + \sqrt{C_0\epsilon^*}\,dW_i'. \end{aligned} \tag{30}$$

Here, dW_i' is another isotropic Wiener process, independent of the one which appears in Eq. (29), and ϵ^* is the modeled dissipation rate. The Generalized Langevin model has parameters G_{ij} and C_0, which jointly provide a model for the fluctuating pressure gradients and the fluctuating velocities.

To complete the Generalized Langevin model, we specify the parameters G_{ij} and C_0, using Durbin's method of elliptic relaxation (Durbin,1991,1993). In this approach the terms which involve fluctuating pressure gradients are modeled with an elliptic equation, by analogy with the fact that the pressure is governed by the Poisson equation. This represents the non-local effect of the wall on the Reynolds-stresses through the fluctuating pressure terms. We introduce a tensor $\wp_{ij}$ to characterize the non-local effect of fluctuating pressure, and set

$$G_{ij} = \frac{\wp_{ij} - \frac{\epsilon^*}{2}\delta_{ij}}{k^*} \quad (31)$$

$$C_0 = \frac{-2\wp_{ij}\left\langle u_i^* u_j^* \right\rangle}{3k^*\epsilon^*}, \quad (32)$$

where k^* is the modeled turbulent kinetic energy. While C_0 is often constant in pdf models, it is chosen here to insure that $\wp_{ij}$ be purely redistributive (Durbin, private communication).

To define $\wp_{ij}$ we first specify time and length scales. Following Durbin, we take the maximum of the turbulent scales and the Kolmogorov scales:

$$T = \sqrt{\left(\frac{k^*}{\epsilon^*}\right)^2 + \left(C_T\sqrt{\frac{\nu}{\epsilon^*}}\right)^2}, \quad (33)$$

$$L = C_L \max\left[\frac{k^{*\frac{3}{2}}}{\epsilon^*}, C_\eta\left(\frac{\nu^3}{\epsilon^*}\right)^{\frac{1}{4}}\right]. \quad (34)$$

Then we specify the non-local term $\wp_{ij}$ with the following elliptic relaxation equation:

$$\left(I - L^2\nabla^2\right)\wp_{ij} = \frac{(1-C_1)\,k^*}{2T}\delta_{ij} + k^* H_{ijkl}\frac{\partial \langle U_k^* \rangle}{\partial x_l} \quad (35)$$

where

$$H_{ijkl} = \left(C_2 + \frac{1}{3}\gamma_5\right)\delta_{ik}\delta_{jl} - \frac{1}{3}\gamma_5\delta_{il}\delta_{jk} + \gamma_5 b_{ik}\delta_{jl} - \gamma_5 b_{il}\delta_{jk}, \quad (36)$$

and b_{ij} is the Reynolds-stress anisotropy tensor. The right hand side of Eq. (35) is the family of stochastic Lagrangian versions of the IP model (Pope,1993) with parameter γ_5, and with IP model constants C_1 and C_2. Far from the wall, $\wp_{ij}$ dominates the Laplacian term on the left hand side of Eq. (35), so the SLW model approaches the IP model. Close to the wall, the Laplacian term becomes important and brings out the non-local response of the pressure fluctuations to the boundary conditions. The complete SLW particle formulation is given by Eqs. (29)-(36).

Pdf and Reynolds-Stress Closures

The model of the previous section can be used to close both Eq. (19) and Eq. (4). The "*" superscript notation is now dropped, with the understanding that all physical variables and pdfs are modeled quantities. The pdf evolution equation which corresponds to the SLW particle equations is

$$\begin{aligned} \frac{\partial f}{\partial t} + V_i\frac{\partial f}{\partial x_i} &= \nu\frac{\partial^2 f}{\partial x_i \partial x_i} + \frac{\partial f}{\partial V_i}\frac{1}{\rho}\frac{\partial \langle \mathcal{P} \rangle}{\partial x_i} \\ +\nu\frac{\partial \langle U_i \rangle}{\partial x_k}\frac{\partial \langle U_j \rangle}{\partial x_k}\frac{\partial^2 f}{\partial V_i \partial V_j} &+ 2\nu\frac{\partial \langle U_j \rangle}{\partial x_i}\frac{\partial^2 f}{\partial x_i \partial V_j} \\ -\frac{\partial}{\partial V_i}\left[G_{ij}\left(V_j - \langle U_j \rangle\right) f\right] &+ \frac{1}{2}C_0\epsilon\frac{\partial^2 f}{\partial V_i \partial V_i}. \end{aligned} \quad (37)$$

The random term in Eq. (29) leads to the important viscous term $\nu\frac{\partial^2 f}{\partial x_i \partial x_i}$ on the right hand side of Eq. (37). The SLW model provides a pdf closure by replacing the last two lines of Eq. (19) with the last two lines of Eq. (37).

To complete the closure at the pdf level, we use Durbin's modeled equation for the dissipation:

$$\begin{aligned} \frac{\tilde{D}\epsilon}{Dt} &= \frac{\partial}{\partial x_i}\left[\left(\nu\delta_{ij} + \frac{C_\mu}{\sigma_\epsilon}\langle u_i u_j \rangle T\right)\frac{\partial \epsilon}{\partial x_j}\right] \\ &+ C_{\epsilon 1}\left(1 + a_1\frac{P}{\epsilon}\right)\frac{P}{T} - C_{\epsilon 2}\frac{\epsilon}{T} \end{aligned} \quad (38)$$

where P is the production of turbulent kinetic energy.

We derive the Reynolds-stress version of the SLW model by taking first and second moments of the pdf equation. The mean velocity equation is derived by multiplying Eq. (37) by V_l and integrating over velocity space. This gives the familiar Reynolds equations exactly:

$$\frac{\tilde{D}\langle U_j \rangle}{Dt} + \frac{\partial}{\partial x_i}\langle u_i u_j \rangle = -\frac{1}{\rho}\frac{\partial \langle \mathcal{P} \rangle}{\partial x_j} + \nu\frac{\partial^2 \langle U_j \rangle}{\partial x_i \partial x_i}. \quad (39)$$

To derive the SLW Reynolds-stress equation, we multiply Eq. (37) by the quantity $(V_l - \langle U_l \rangle)(V_m - \langle U_m \rangle)$ and integrate over velocity space to give

$$\begin{aligned} \frac{\tilde{D}\langle u_i u_j \rangle}{Dt} &= T_{(v)ij} + T_{(t)ij} + P_{ij} \\ &+ G_{ik}\langle u_k u_j \rangle + G_{jk}\langle u_k u_i \rangle + C_0\epsilon\delta_{ij}. \end{aligned} \quad (40)$$

By comparing Eq. (40) with (4), we see that the SLW model uses the Generalized Langevin terms for the combined effects of fluctuating velocity - pressure gradient correlations and dissipation:

$$\phi_{ij} - \epsilon_{ij} = G_{ik}\langle u_k u_j \rangle + G_{jk}\langle u_k u_i \rangle + C_0\epsilon\delta_{ij}. \quad (41)$$

At the Reynolds-stress level, we also need a model for the turbulent transport term $T_{(t)ij} = \frac{\partial \langle u_i u_j u_k \rangle}{\partial x_k}$. Following Durbin, we use a gradient diffusion model:

$$T_{(t)ij} = \frac{\partial}{\partial x_k}\left(\frac{C_\mu}{\sigma_k}\langle u_k u_l \rangle T\frac{\partial \langle u_i u_j \rangle}{\partial x_l}\right). \quad (42)$$

The complete Reynolds-stress closure is specified by Eqs. (31) - (36), (38) - (40), and (42). Model constants are

$$\begin{gathered} C_1 = 1.85;\ C_2 = 0.25;\ C_\mu = 0.23;\ \sigma_k = 1.2;\ \gamma_5 = 0.12; \\ \sigma_\epsilon = 1.65;\ C_{\epsilon 1} = 1.44;\ C_{\epsilon 2} = 1.9;\ a_1 = 0.1; \\ C_T = 6.0;\ C_L = 0.123;\ C_\eta = 72.0. \end{gathered} \quad (43)$$

Boundary Conditions and Near-wall Behavior

At the wall, we impose the no slip condition on the mean and fluctuating velocities, and we set $\wp_{ij}$ as follows:

$$\langle U_i \rangle = \langle u_i u_j \rangle = 0, \quad (44)$$

$$n_i\frac{\partial k}{\partial x_i} = 0, \quad (45)$$

$$\wp_{ij} = -4.5\,\epsilon\, n_i n_j, \quad (46)$$

where $\mathbf{n}$ is the wall normal. Because the only non-zero component of $\wp_{ij}$ at the wall is in the normal direction, the model can be implemented in more complicated geometries without additional assumptions about the direction of the mean flow.

To leading order, the SLW near-wall Reynolds-stress equations and their solutions can be determined analytically, using a similar argument to the one which appears for the exact equations. Close to the wall, the SLW Reynolds-stress equations become

$$\nu\frac{\partial^2\langle u_1^2\rangle}{\partial y^2} - \frac{\epsilon}{k}\langle u_1^2\rangle = 0, \tag{47}$$

$$\nu\frac{\partial^2\langle u_2^2\rangle}{\partial y^2} + \frac{\frac{4}{3}\wp_{22}-\epsilon}{k}\langle u_2^2\rangle = O(y), \tag{48}$$

$$\nu\frac{\partial^2\langle u_3^2\rangle}{\partial y^2} - \frac{\epsilon}{k}\langle u_3^2\rangle = 0, \tag{49}$$

$$\nu\frac{\partial^2\langle u_1u_2\rangle}{\partial y^2} + \frac{\wp_{22}-\epsilon}{k}\langle u_1u_2\rangle = O(y), \tag{50}$$

with solutions

$$\langle u_1^2\rangle \sim y^2, \tag{51}$$

$$\langle u_2^2\rangle \sim y^3, \tag{52}$$

$$\langle u_3^2\rangle \sim y^2, \tag{53}$$

$$\langle u_1u_2\rangle \sim y^3. \tag{54}$$

It is clear from comparing Eqs. (47) - (50) with Eqs. (14) - (17) that the SLW model reproduces the dominant balance of transport with dissipation in the near-wall Reynolds-stresses exactly in the tangential directions. To leading order, this aspect of the model captures the near-wall anisotropy. Near-wall inhomogeneity is also captured with the appearance of the exact viscous transport terms on the left hand side of Eqs. (47) - (50). Comparison of Eqs. (51) - (54) with Eqs. (9) - (12) shows that all Reynolds-stresses scale correctly with y near the wall, except for $\langle u_2^2\rangle$, which approaches 0 as y^3 rather than the correct rate of y^4. But the important processes (those of order 1) are the portions of viscous transport and dissipation which occur only in the (1,1), and (3,3) directions near the wall; this analysis shows that the SLW model represents those balances correctly. So before considering any calculations, we know that the SLW model can characterize important aspects of near-wall flows.

FULLY DEVELOPED CHANNEL FLOW

The SLW model is tested for fully developed channel flow with $Re = 395$, based on the friction velocity and the channel halfwidth. The Reynolds-stress equations are discretized on a 150 cell grid, and solved with Newton's method using a fully implicit scheme. SLW model predictions of velocity and Reynolds-stresses are shown in Figures (1) and (2), together with DNS data of Kim. Model predictions of Reynolds-stress budgets up to $y^+ = 200$ are shown and compared to DNS data of Mansour in figures (3) - (6). All

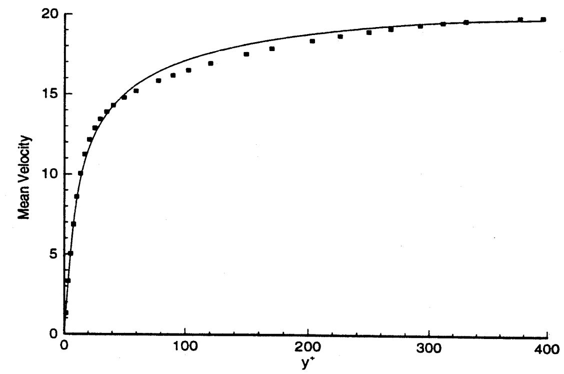

Figure 1: Mean velocity: comparison of SLW model (line) with DNS data (symbols)

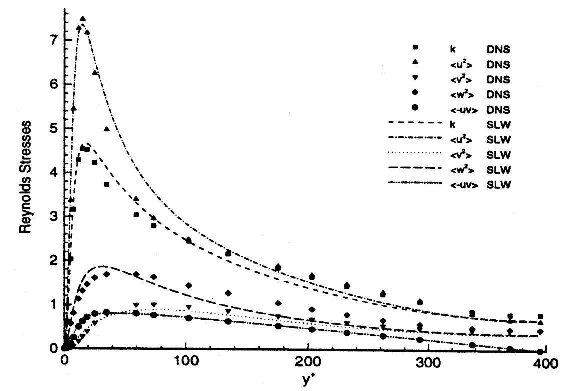

Figure 2: Reynolds-stresses: comparison of SLW model (line) with DNS data (symbols)

velocities shown are normalized with the friction velocity and plotted against the distance from the wall in wall units. It should be understood that the SLW model provides an expression for the sum of pressure correlation and dissipation terms in Eq. (41), but not for each term separately. In the budgets shown, Rotta's model for anisotropic dissipation,

$$\epsilon_{ij} = \langle u_iu_j\rangle\frac{\epsilon}{k} \tag{55}$$

is used to distinguish the model predicted values of ϕ_{ij} from those of ϵ_{ij}. Generally good agreement with DNS is achieved with the SLW model, except for the near-wall budget of $\langle u_2^2\rangle$, in which the balance between pressure correlations and dissipation is underpredicted. This occurs because the Rotta's dissipation model is wrong for this component; we can infer from Table (1) that very close to the wall the dissipation in the (2,2) direction is 4 times as large as Eq. (55) predicts. Overall, these results show that elliptic relaxation is a viable technique for near-wall modeling with the pdf approach. For future work, the SLW model will be solved as a pdf model using a Monte Carlo code, and it will be tested with more complex wall bounded turbulent flows.

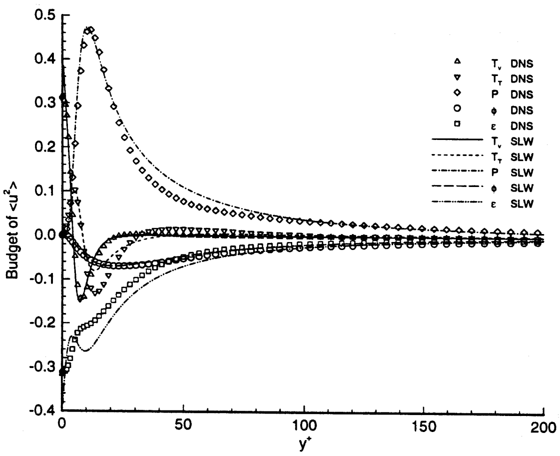

Figure 3: Budget of $\langle u_1^2 \rangle$: comparison of SLW model (line) with DNS data (symbols)

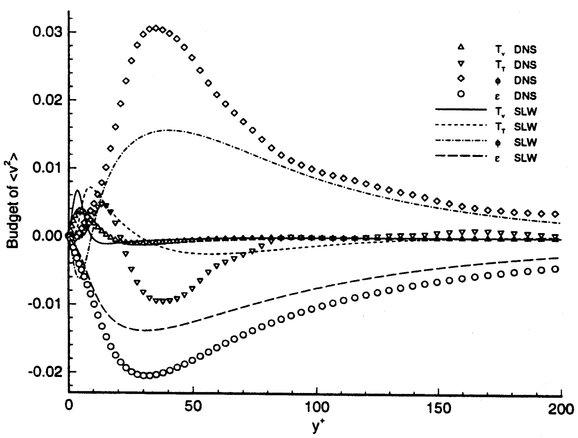

Figure 4: Budget of $\langle u_2^2 \rangle$: comparison of SLW model (line) with DNS data (symbols)

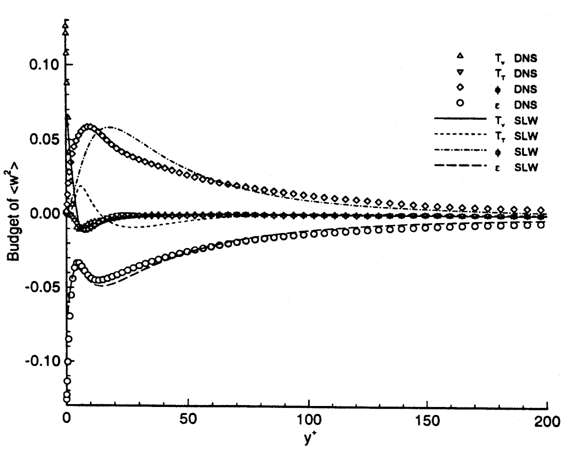

Figure 5: Budget of $\langle u_3^2 \rangle$: comparison of SLW model (line) with DNS data (symbols)

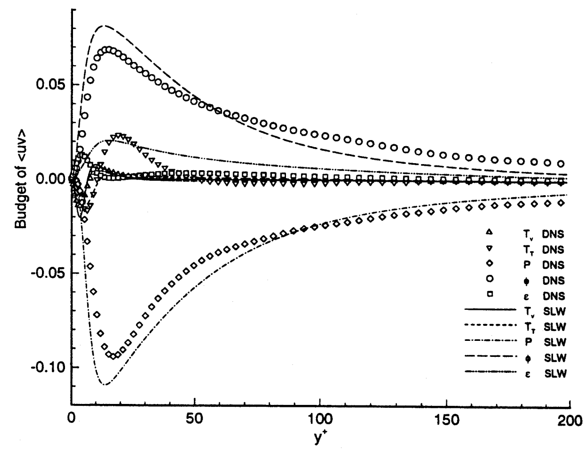

Figure 6: Budget of $\langle u_1 u_2 \rangle$: comparison of SLW model (line) with DNS data (symbols)

REFERENCES

Durbin, P., 1993, "A Reynolds-Stress Model for Near-Wall Turbulence," *Journal of Fluid Mechanics*, Vol. 249, pp. 465-498.

Durbin, P., 1991, "Near-Wall Turbulence Closure Modeling Without 'Damping Functions'," *Theoretical and Computational Fluid Dynamics*, Vol. 3, pp. 1-13.

Einstein, A., 1926, *Investigations on the Theory of the Brownian Movement*, Methuen and Co. Ltd., London, England, Translation by A. D. Cowper, Edited by R. Furth.

Haworth, D. C. and Pope, S. B., 1986, "A Generalized Langevin Model for Turbulent Flows," *Physics of Fluids A*, Vol. 29, pp. 387-405.

Haworth, D. C. and Pope, S. B., 1987, "A Pdf Modeling Study of Self-Similar Turbulent Free Shear Flows," *Physics of Fluids A*, Vol. 30, pp. 1026-1044.

Launder, B. E., Reece, G. J. and Rodi, W., 1975, "Progress in the Development of a Reynolds-stress Turbulence Closure," *Journal of Fluid Mechanics*, Vol. 68, pp. 537-566.

Mansour, N. N., Kim, J. and Moin, P., 1988, "Reynolds-stress and Dissipation-Rate Budgets in a Turbulent Channel Flow," *Journal of Fluid Mechanics*, Vol. 194, pp. 15-44.

Naot, D., Shavit, A. and Wolfshtein, M., 1970, "Interactions Between Components of the Turbulent Velocity Correlation Tensor," *Isreal Journal of Technology*, Vol. 8, pp. 259-269.

Pope, S. B., 1993, "On the Relationship Between Stochastic Lagrangian Models of Turbulence and Second Moment Closures," *Physics of Fluids*, Vol. 6, pp. 973-985.

Pope, S. B., 1985, "Pdf Methods for Turbulent Reactive Flows," *Progress in Energy and Combustion Science*, Vol. 11, pp. 119-192.

Wax, N., 1954, *Noise and Stochastic Processes*, Dover Publications inc., New York, NY.