Diffusion Behind a Line Source in Grid Turbulence

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Abstract

The flow considered is the thermal wake downstream of a fine heated wire in grid turbulence. The inability of generally-applicable turbulent diffusion models and second-order closures to calculate the mean and variance of the temperature is reviewed. Calculations based on the transport equation for the joint probability density function (pdf) of velocity and temperature show good agreement with measurements of mean temperature but not with those of the variance. Better calculations of the variance are obtained with a method based on the joint pdf conditional on the lateral velocity at the source.

Introduction

The thermal wake downstream of a heated wire in grid turbulence has been the subject of several experimental [1–4] and theoretical studies ([5–10], for example). The flow is of fundamental theoretical importance and is also relevant to the practical problem of the dispersion of heat and pollutants in the environment [11].

In comparison to free shear flows – jets, wakes, mixing layers – the thermal wake appears simple: the mean velocity is uniform and the turbulence is isotropic. But turbulence models that are applicable to free shear flows (e.g. $k - \varepsilon$ and Reynolds-stress models) produce qualitatively incorrect results for the thermal wake [9]. This is because the thermal wake thickness σ is, initially, much smaller than the integral length scale, and because we are interested in the wake at convection times that are much smaller than the integral time scale of the turbulence.

In this work, the thermal wake is studied using velocity-temperature joint pdf transport equations [12, 13]. It is found that the standard (unconditional) pdf method is successful in calculating the mean temperature field, but not the temperature variance. A second method – based on the joint pdf conditional on the lateral velocity at the heated wire – is successful in calculating the normalized variance profiles and calculates the absolute magnitude of the variance to within a factor of two. This level of accuracy is comparable with that obtained using two-particle dispersion models [4].

Thermal Wake

The thermal wake is sketched on Fig. 1. A fine heated wire of diameter d is placed normal to the flow, a distance x_0 downstream of a turbulence-generating grid of mesh size M. The mean velocity U is uniform and the turbulence intensity normal to the flow and the wire is found to vary according to the power law:

$$\langle w^2 \rangle / U^2 = A(x/M)^{-m}. \tag{1}$$

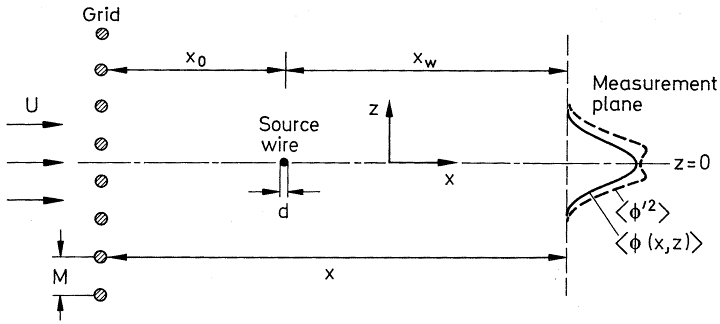

Fig. 1. Sketch of a heated wire downstream of a turbulence generating grid

In the measurements of Warhaft [3] – with which the calculations are compared – the conditions are: $U = 7.0$ m/s, $M = 25$ mm, $x_0/M = 52$, $A = 0.0714$, $m = 1.32$, $d = 0.025$ mm (for measurements up to $x_w/M = 8.1$), and $d = 0.127$ mm (for measurements beyond $x_w/M = 8.1$).

The wire is intended to be sufficiently fine that it does not significantly affect the mean or fluctuating velocity field. Although the wire is quite hot, the excess temperature rapidly falls to a few degrees Celsius. Thus, except in the immediate vicinity of the wire, the excess temperature is a conserved, passive scalar.

The principal quantities of interest are the mean $\langle\phi\rangle$ and variance $\langle\phi'^2\rangle$ of the normalized excess temperature $\phi(\underset{\sim}{x}, t)$. (Angled brackets and primes denote means and fluctuations about the mean, respectively.) At any distance x_w downstream of the source (i.e. the heated wire), the profile of $\langle\phi\rangle$ is found to be Gaussian [1–4],

$$\langle\phi(x_w, z)\rangle = (\sigma\sqrt{2\pi})^{-1} \exp(-\tfrac{1}{2}z^2/\sigma^2). \tag{2}$$

The integral $\int_{-\infty}^{\infty} \langle\phi(x_w, z)\rangle\, dz$ is a conserved quantity, and ϕ is normalized so that this integral is unity. Profiles of $\langle\phi^2\rangle$ (but not of $\langle\phi'^2\rangle$) also appear to be Gaussian [4].

There is an analogy [1] between the thermal wake and the temperature field resulting from the instantaneous production (at time $t = 0$) of an excess temperature distribution along the plane $z = 0$. Thermal wake statistics at (x_w, z) are similar to those at (t, z) resulting from the plane source, where $t = x_w/U$. The analogy depends upon the turbulence intensity $\langle u^2\rangle^{1/2}/U$ being small and mean axial gradients being small in comparison with lateral gradients. In common with previous work, our calculations are of the statistically one-dimensional, time dependent evolution of the temperature field from the plane source in decaying homogeneous turbulence. The results at time t are compared with the thermal wake data of Warhaft at $x_w = Ut$. (Henceforth t is used interchangeably with x_w/U.)

Previous Work

Mean Temperature

A theory should predict a Gaussian mean temperature profile (Eq. 2) and should determine the evolution of the width of the profile $\sigma(t)$. In high Reynolds-number, non-decaying, homogeneous turbulence, there are three regimes:

Molecular diffusion, $t \ll \Gamma/\langle w^2 \rangle$: $\sigma^2 \approx 2\Gamma t$.
Turbulent convection, $\Gamma/\langle w^2 \rangle \ll t \ll T$: $\sigma^2 \approx \langle w^2 \rangle t^2$.
Turbulent diffusion, $t \gg T$: $\sigma^2 \approx 2\Gamma_t t$.

Here Γ is the thermal diffusivity, T is the Lagrangian integral time scale, and Γ_t is the turbulent diffusion coefficient. Taylor [5] neglected molecular diffusion and obtained an expression for σ in terms of the Lagrangian velocity autocorrelation function. This expression (which can readily be generalized to the case of decaying turbulence [4]) correctly accounts for the turbulent convection and diffusion regimes. The effect of molecular diffusion is simply to increase σ^2 by $2\Gamma t$ [2].

The exact equation for the mean of ϕ is

$$\frac{\partial \langle \phi \rangle}{\partial t} = \Gamma \frac{\partial^2 \langle \phi \rangle}{\partial z^2} - \frac{\partial \langle w\phi \rangle}{\partial z}. \tag{3}$$

Turbulence models attempt to approximate the scalar flux $\langle w\phi \rangle$ so that Eq. 3 can be solved for $\langle \phi \rangle$.

Turbulent diffusion models (mixing length, $k - \varepsilon$ etc.) approximate the scalar flux by

$$\langle w\phi \rangle = -\Gamma_t \frac{\partial \langle \phi \rangle}{\partial z}. \tag{4}$$

As their name implies, these models are applicable only to the turbulent diffusion regime: early on ($t \ll T$) they predict far too rapid spreading. For example, according to the $k - \varepsilon$ model (see e.g. [14]) the turbulent diffusion coefficient is

$$\Gamma_t = \frac{C_\mu}{\sigma_f} k^2/\varepsilon, \tag{5}$$

where k is the turbulent kinetic energy ($k = 3/2 \langle w^2 \rangle$ in isotropic turbulence) and ε is the rate of dissipation

$$\varepsilon = -dk/dt. \tag{6}$$

The model constants are taken to be $C_\mu = 0.09$ and $\sigma_f = 0.7$. With the turbulence decaying according to Eq. 1, the $k - \varepsilon$ model yields (neglecting molecular diffusion)

$$\sigma^2 = \int_0^t 2\Gamma_t(t')\, dt' = \frac{3AM^2 C_\mu}{m(2-m)\sigma_f} \left\{ \left(\frac{x}{M}\right)^{(2-m)} - \left(\frac{x_0}{M}\right)^{(2-m)} \right\}. \tag{7}$$

On Fig. 2, the broken line shows this predicted evolution of σ compared to Warhaft's data (triangles). It may be seen that Eq. 7 is qualitatively and quantitatively incorrect. At the first measurement station, the calculated width is over three times that measured.

Turbulent diffusion models can be made to work, by making Γ_t an explicit function of σ or t [9, 10]. But such ad hoc modifications are inconsistent with the principles of invariant modelling [15, 16] and have little general utility.

Second-order closures experience the same problem [7, 9]. The correct evolution of σ can be obtained only by modifying the transport coefficient for the third moment $\langle w^2 \phi \rangle$ to be a function of the temperature field.

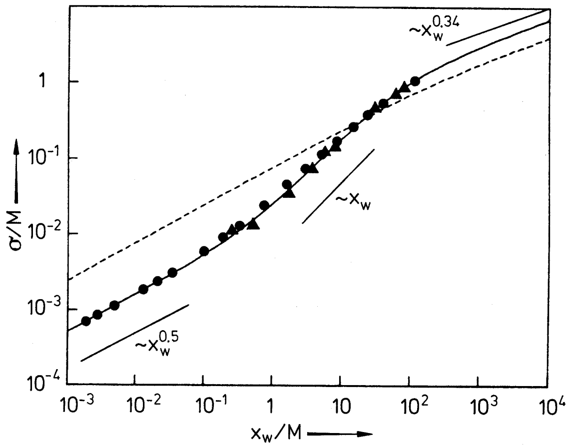

Fig. 2. Thermal wake thickness against distance from the wire. ---- $k-\varepsilon$ model, ● unconditional pdf method, ——— conditional pdf method, ▲ experimental data [3]

Temperature Variance

The exact equation for the variance of ϕ is

$$\frac{\partial\langle\phi'^2\rangle}{\partial t}+\frac{\partial\langle w\phi'^2\rangle}{\partial z}=\Gamma\frac{\partial^2\langle\phi'^2\rangle}{\partial z^2}-2\langle w\phi'\rangle\frac{\partial\langle\phi\rangle}{\partial z}-2\varepsilon_\phi, \tag{8}$$

where the scalar dissipation is

$$\varepsilon_\phi\equiv\Gamma\left\langle\frac{\partial\phi'}{\partial x_i}\frac{\partial\phi'}{\partial x_i}\right\rangle. \tag{9}$$

The simplest modelling assumption for ε_ϕ is that the scalar-dissipation time scale $1/2\langle\phi'^2\rangle/\varepsilon_\phi$ is proportional to the mechanical-dissipation time scale k/ε: then ε_ϕ is given by

$$\varepsilon_\phi=C_\phi\tfrac{1}{2}\langle\phi'^2\rangle\,\varepsilon/k. \tag{10}$$

The data of Warhaft and Lumley [17] suggest that C_ϕ is not a universal constant, but a value of $C_\phi=2.0$ is commonly employed (e.g. [12]).

The validity of Eq. 10 for the thermal wake can be tested rather directly. Let $I(t)$ be the integral of the scalar variance:

$$I\equiv\int_{-\infty}^{\infty}\langle\phi'^2\rangle\,dz. \tag{11}$$

Then, from Eqs. 3 and 8 and the knowledge that the profile of $\langle\phi\rangle$ is Gaussian, we obtain

$$\frac{dI}{dt}=\bar{P}-C_\phi\frac{\varepsilon}{k}I, \tag{12}$$

where

$$\bar{P}\equiv\left(\frac{1}{2}\frac{d\sigma^2}{dt}-\Gamma\right)\Big/(2\sigma^3\sqrt{\pi}). \tag{13}$$

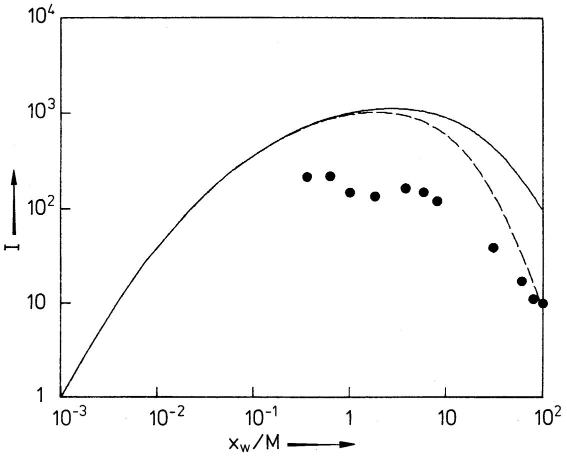

Fig. 3. Integral variance against distance from the wire. ——— unconditional model (Eqs. 10–13) $C_\phi = 2.0$, – – – unconditional model $C_\phi = 4.0$, ● from experimental data [3]

It may be noted that a knowledge of $\sigma(t)$ is sufficient to determine the production term $\bar{P}$. In the section after next, a model that accurately determines σ is described (see the solid line on Fig. 2). Using this result (Eq. A 20) $\bar{P}(t)$ was evaluated and then Eq. 12 was integrated numerically to yield $I(t)$. The result is compared with Warhaft's data on Fig. 3.

It may be seen from Fig. 3 that with the conventional value $C_\phi = 2.0$, the calculated integral variance I exceeds the measured value by, typically, a factor of 7 to 12. With the higher value $C_\phi = 4.0$, this range is reduced to 4 to 7. Since $\sigma(t)$ given by Eq. A 20 agrees extremely well with the data (see Fig. 2), the only significant assumption in the analysis is the modelling of the scalar dissipation, Eq. 10. The results shown on Fig. 3 clearly indicate, therefore, that this model is grossly in error for the thermal wake. If ε_ϕ were determined instead from a consistently-modelled transport equation [16] the result would be little different.

In light of these observations, Sykes, Lewellen and Parker [7] replaced Eq. 10 with a model for ε_ϕ in terms of a scalar length scale, for which a modelled ordinary differential equation was solved. They were then able to obtain accurate calculations of the scalar variance, including the effect of the source size. While this method succeeded for the case considered, because of the way in which both $\langle w^2 \phi \rangle$ and ε_ϕ are modelled, the model does not appear to be applicable to the general case (i.e. more than one transported scalar [16], or more than one source [7]).

A completely different method of calculating the temperature variance is based on the relative dispersion of two fluid particles or, better, of two molecules. There has been considerable work on this method recently (see [8] for references) culminating in the comparison of theory and experiment by Stapountzis, Sawford, Hunt and Britter [4]. The mean thermal wake width $\sigma(t)$ is calculated to within experiment error; the variance is calculated to within a factor of two; and normalized variance profiles agree well with the data.

Unconditional pdf Method

Turbulent diffusion models are unsuccessful in calculating the mean wake thickness $\sigma(t)$ because the second moment $\langle w \phi' \rangle$ is approximated by a gradient diffusion model. Similarly, in second-order closures a gradient-diffusion model is used for the third moment $\langle w^2 \phi \rangle$ and consequently σ is not calculated correctly. In this instance gradient-diffusion models fail because the length scale of variation of the temperature field is much smaller than the turbulence integral scale. But gradient-diffusion models can be entirely avoided by making use of the velocity-scalar joint pdf equation.

For a general flow, $f(\boldsymbol{V}, \psi; \boldsymbol{x}, t)$ is the probability density of the joint events $\boldsymbol{U}(\boldsymbol{x}, t) = \boldsymbol{V}$ and $\phi(\boldsymbol{x}, t) = \psi$. The four new independent variables V_1, V_2, V_3, ψ form the velocity-temperature space. An exact equation for the joint pdf can be derived [12]: for the flow under consideration, the equation for $f(\boldsymbol{V}, \psi; z, t)$ is

$$\frac{\partial f}{\partial t} + V_3 \frac{\partial f}{\partial z} = -\frac{\partial}{\partial \psi}\{f \langle \Gamma \nabla^2 \phi \,|\, \boldsymbol{V}, \psi \rangle\} - \frac{\partial}{\partial V_i}\{f \langle a_i \,|\, \boldsymbol{V}, \psi \rangle\}, \tag{14}$$

where, for any function Q, $\langle Q \,|\, \boldsymbol{V}, \psi \rangle$ is the expectation of Q conditional upon the joint events $\boldsymbol{U} = \boldsymbol{V}$ and $\phi = \psi$. The force (per unit mass) on a fluid particle is

$$a_i = \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j} - \frac{1}{\varrho} \frac{\partial p}{\partial x_i}, \tag{15}$$

where ν is the kinematic viscosity, p the pressure, and ϱ the density.

In the joint pdf equation only the terms on the right-hand side need to be modelled. It may be seen that the convective term $V_3 \, \partial f / \partial z$ is in closed form and hence no transport model is required. Stochastic models for the terms in $\nabla^2 \phi$ and a_i have been described previously [12, 13]. The joint pdf equation incorporating these models can be integrated to form transport equations for the scalar flux and scalar variance:

$$\begin{aligned} &\frac{\partial}{\partial t}\langle w\phi\rangle + \frac{\partial}{\partial z}\langle w^2 \phi' \rangle + \langle w^2 \rangle \frac{\partial}{\partial z}\langle \phi \rangle - \Gamma \frac{\partial^2}{\partial z^2}\langle w\phi \rangle \\ &\qquad = -C_m \frac{\varepsilon}{k}\langle w\phi\rangle, \end{aligned} \tag{16}$$

$$\begin{aligned} &\frac{\partial}{\partial t}\langle \phi'^2\rangle + \frac{\partial}{\partial z}\langle w \phi'^2 \rangle + 2\langle w\phi \rangle \frac{\partial}{\partial z}\langle \phi \rangle - \Gamma \frac{\partial^2}{\partial z^2}\langle \phi'^2 \rangle \\ &\qquad = -C_\phi \frac{\varepsilon}{k}\langle \phi'^2\rangle. \end{aligned} \tag{17}$$

These equations illustrate the effect of the model constants which were chosen to be $C_m = 2.9$ and $C_\phi = 2.0$.

The modelled joint pdf equation was solved by a Monte Carlo method [13] for the conditions of Warhaft's experiment. The profiles of $\langle \phi \rangle$ were found to be Gaussian and the development of $\sigma(t)$ (circles on Fig. 2) is in excellent agreement with the data. In high-Reynolds number decaying turbulence ($\langle w^2 \rangle \propto x_w^{-m}$), σ can be expected to vary as $\sqrt{x_w}$, x_w, and $x_w^{(1-m/2)}$, respectively, in the three regimes – molecular diffusion, turbulent convection, turbulent diffusion. The calculations exhibit the correct variations in the first two regimes, and are consistent with the third regime. Neither the calculations nor the measurements extend far enough to show the third regime distinctly.

It may be seen that the equation for the variance derived from the modelled pdf equation (Eq. 17) is just the exact equation (Eq. 8) with the scalar dissipation modelled according to Eq. 10. Consequently, the integrated variance $I(t)$ obtained from the pdf equation is the same as that obtained previously (Eqs. 12–13, Fig. 3), and is greater than the measured value by, typically, a factor of 7–12.

Thus this unconditional pdf method is successful in calculating the mean spreading, but the calculated variance is grossly in error.

Conditional pdf Method

In order to obtain more accurate calculations of the temperature variance, we seek a more complete statistical description of the thermal wake. In the early stages, the random variable that has most influence upon the wake's development is w_0 – the lateral velocity at the source at time zero:

$$w_0 \equiv w(\boldsymbol{x} = 0,\ t = 0). \tag{18}$$

Indeed, at very early times, the temperature profile is completely determined by w_0. It is a laminar wake of thickness $\sqrt{2\Gamma t}$ that has been convected laterally a distance $w_0 t$.

We shall therefore consider statistical quantities conditioned on $w_0 = \tilde{w}$ – primarily $\tilde{f}(\boldsymbol{V}, \psi \mid \tilde{w}; \boldsymbol{x}, t)$, the joint probability of $\boldsymbol{U} = \boldsymbol{V}$ and $\phi = \psi$, conditional upon $w_0 \equiv w(\boldsymbol{x} = 0,\ t = 0) = \tilde{w}$. The unconditional joint pdf (at any $\boldsymbol{x}, t$) can readily be recovered by

$$f(\boldsymbol{V}, \psi) = \int_{-\infty}^{\infty} \tilde{f}(\boldsymbol{V}, \psi \mid \tilde{w})\, g(\tilde{w})\, d\tilde{w}, \tag{19}$$

where $g(\tilde{w})$ is the pdf of w_0 which (in grid turbulence) is known to be Gaussian [18]:

$$g(\tilde{w}) = \frac{1}{\sqrt{2\pi \langle w_0^2 \rangle}} \exp(-\tfrac{1}{2}\tilde{w}^2 / \langle w_0^2 \rangle). \tag{20}$$

The conditional mean profile is

$$\tilde{\phi} \equiv \langle \phi \mid \tilde{w} \rangle = \iint \psi\, \tilde{f}(\boldsymbol{V}, \psi \mid \tilde{w})\, d\boldsymbol{V}\, d\psi, \tag{21}$$

and, again the unconditional mean is recovered by

$$\langle \phi \rangle = \int \langle \phi \mid \tilde{w} \rangle\, g(\tilde{w})\, d\tilde{w}. \tag{22}$$

(Here, and henceforth, the limits of integration $\pm\infty$ are not explicitly indicated. Note also that $\tilde{\phi}$ and ϕ'' (defined below) depend on $\tilde{w}$ as well as on $\boldsymbol{x}, t$.)

The principal virtue of this approach is that fluctuations in ϕ can be decomposed into two parts: the difference between conditional and unconditional means $(\tilde{\phi} - \langle \phi \rangle)$, and fluctuations about the conditional mean $\phi'' \equiv \phi - \tilde{\phi}$. The unconditional variance can then be expressed as

$$\langle \phi'^2 \rangle = \int g(\tilde{w})\, (\tilde{\phi} - \langle \phi \rangle)^2\, d\tilde{w} + \int g(\tilde{w})\, \widetilde{\phi''^2}\, d\tilde{w}, \tag{23}$$

where the conditional variance is

$$\widetilde{\phi''^2} \equiv \langle \phi''^2 \mid \tilde{w} \rangle = \langle (\phi - \langle \phi \mid \tilde{w} \rangle)^2 \mid \tilde{w} \rangle. \tag{24}$$

The first contribution to $\langle \phi'^2 \rangle$ is due to the meandering or flapping of the wake [6, 4]. The second contribution is due to fluctuations about the conditional mean.

The exact transport equation for $\tilde{f}$ is

$$\frac{\partial \tilde{f}}{\partial t} + V_i \frac{\partial \tilde{f}}{\partial x_i} = -\frac{\partial}{\partial \psi}\{\tilde{f}\langle \Gamma \nabla^2 \phi \,|\, V, \psi, \tilde{w}\rangle\} - \frac{\partial}{\partial V_i}\{\tilde{f}\langle a_i \,|\, V, \psi, \tilde{w}\rangle\}. \tag{25}$$

Since the conditional joint pdf contains two-time Lagrangian information, the modelling of the conditional expectations should be consistent with our knowledge of such statistics [19]. The following model, which is based on Langevin's equation [20], is consistent with Kolmogorov's inertial range scaling laws [22]:

$$\langle a_i \,|\, V, \psi, \tilde{w}\rangle = -\frac{1}{\varrho}\frac{\partial}{\partial x_i}\langle p \,|\, V, \psi, \tilde{w}\rangle - \left(\tfrac{1}{2} + \tfrac{3}{4} C_0\right)\frac{\varepsilon}{k}\{V_i - \langle U_i \,|\, \tilde{w}\rangle\} - \tfrac{1}{2} C_0 \varepsilon \frac{\partial \ln \tilde{f}}{\partial V_i}. \tag{26}$$

And, according to Kolmogorov's second hypothesis, C_0 is a universal constant.

For the scalar dissipation term, the model

$$-\frac{\partial}{\partial \psi}\{\tilde{f}\langle \Gamma \nabla^2 \phi \,|\, V, \psi, \tilde{w}\rangle\} = \Gamma \nabla^2 \tilde{f} + \tilde{\varepsilon}_\phi \frac{\partial}{\partial \psi}\{\tilde{f}(\psi - \tilde{\phi})\}/\widetilde{\phi''^2}, \tag{27}$$

is exact for the mean and variance (although it may be inaccurate for higher moments). And, although it proved inapplicable in the unconditional case, we hypothesize that the conditional scalar dissipation can be modelled by:

$$\tilde{\varepsilon}_\phi \equiv \left\langle \frac{\partial \phi''}{\partial x_i}\frac{\partial \phi''}{\partial x_i} \middle|\, \tilde{w} \right\rangle = C_\phi \tfrac{1}{2} \widetilde{\phi''^2}\, \varepsilon / k. \tag{28}$$

In this conditional case, $\tilde{\varepsilon}_\phi$ is due solely to turbulent straining (and not to flapping) and so may be expected to scale according to Eq. 28.

Equations 25–28 could be solved (for given $\tilde{w}$) by a Monte Carlo method to yield $\tilde{f}$. Then the unconditional pdf could be obtained from Eq. 19. Instead of following this approach, we obtain (in the Appendix) approximate analytical solutions.

The conditional mean profile of ϕ is Gaussian

$$\langle \phi \,|\, \tilde{w}\rangle = \frac{1}{\sqrt{2\pi}\,\tilde{\sigma}} \exp\left(-\tfrac{1}{2}(z - \tilde{z})^2/\tilde{\sigma}^2\right). \tag{29}$$

Expressions are given in the Appendix for the conditionally-expected location of the center of the wake $\tilde{z}(t)$ and its width $\tilde{\sigma}$. The unconditional wake thickness σ, obtained from Eqs. 22 and 29, is shown as the full line on Fig. 2. Even though the constant value $C_0 = 2.083$ was chosen with respect to this data, the agreement over the whole range of x_w/M lends support to the modelling. (C_0 was not determined to four figures; rather the quantity $(3\,C_0)^{1/2} = 2.5$ was determined to two figures.)

The experimental data shown on Fig. 2 are for one set of conditions in Warhaft's experiment (i.e. $x_0/M = 52$, $U = 7.0$ m/s). It is shown in the Appendix that the theory

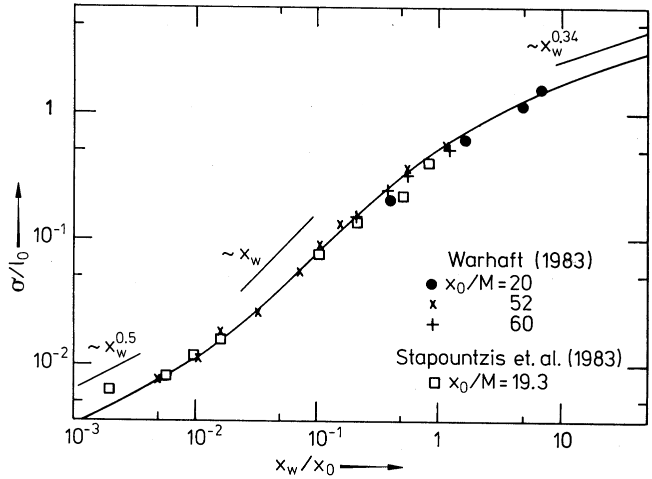

Fig. 4. Thermal wake thickness (σ) normalized by the turbulent length scale at the wire (l_0) against downstream distance from the wire (x_w) normalized by the distance of the wire from the grid (x_0). The solid line is from theory (Eq. A 20) and the symbols represent various experimental data

predicts that (except in the molecular diffusion regime) σ/l_0 is a unique function of x_w/x_0, where l_0 is the length scale $k^{3/2}/\varepsilon$ at the wire. Figure 4 shows σ/l_0 plotted against x_w/x_0 for the three values of x_0/M considered by Warhaft, and the single value of x_0/M considered by Stapountzis et al. It may be seen that the theory is in excellent agreement with the data which do, indeed, collapse in this plot.

We now turn our attention to the determination of the unconditional variance $\langle\phi'^2\rangle$. As previously mentioned, Eq. 23, $\langle\phi'^2\rangle$ can be decomposed into a contribution from the wake flapping

$$\langle\phi'^2\rangle_{\text{flap}} \equiv \int g(\tilde{w})\,(\tilde{\phi} - \langle\phi\rangle)^2\, d\tilde{w}, \tag{30}$$

and a contribution from fluctuations about the conditional mean

$$\langle\phi'^2\rangle_c \equiv \int g(\tilde{w})\,\widetilde{\phi''^2}\, d\tilde{w}, \tag{31}$$

Similarly, the integral variance I, Eq. 11, can be decomposed as

$$I = I_{\text{flap}} + I_c, \tag{32}$$

where I_{flap} and I_c are the integrals over z, of $\langle\phi'^2\rangle_{\text{flap}}$ and $\langle\phi'^2\rangle_c$.

Since expressions for both $\langle\phi\rangle$ and $\tilde{\phi}$ have been obtained, $\langle\phi'^2\rangle_{\text{flap}}$ can be determined from Eq. 30, and I_{flap} by integrating the result: the results are given in the Appendix. The conditional integral variance I_c can be determined in the same way as I was determined for the unconditional case. It satisfies Eq. 12 (with I_c replacing I), with the production $\bar{P}$ given by Eq. 13 with $\tilde{\sigma}$ replacing σ (see Appendix). This equation was integrated numerically to determine I_c. I was then determined from Eq. 32.

The calculated integral variance I and the flapping contribution I_{flap} are shown on Fig. 5 for two values of the model constant C_ϕ. It may be seen that up to $x_w/M \approx 1.0$, I_{flap} contributes more than 90% of the variance, whereas beyond $x_w/M \approx 10$ it contributes less than 25%. At the first two measurement locations ($x_w/M = 0.36$ and 0.62) the calculations are in excellent agreement with the data. By $x_w/M = 8.1$, where the influence of C_ϕ begins to appear, the calculated variance is about twice the measured value. With $C_\phi = 2.0$, the calculated variance is four times the measured value at $x_w/M = 100$, whereas with $C_\phi = 4.0$,

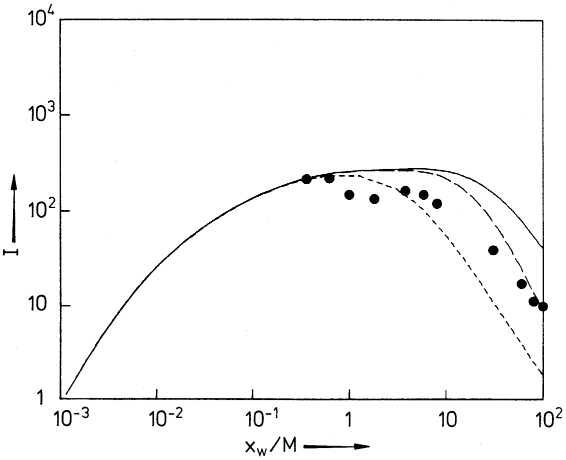

Fig. 5. Integral variance against distance from the wire. ——— conditional pdf method $C_\phi = 2.0$, — — — conditional pdf method $C_\phi = 4.0$, - - - - flapping contribution I_{flap}, ● from experimental data [3]

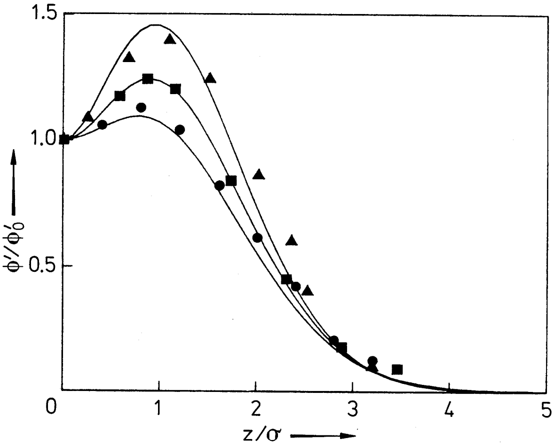

Fig. 6. Normalized standard deviation against cross-stream distance. ——— conditional pdf method. ▲, ■, ● data [3] at $x_w/M = 0.36, 0.62, 1.00$, respectively

the calculations never exceed the measurements by more than a factor of two. This level of agreement (at least with $C_\phi = 4.0$) is comparable with that obtained by Stapountzis et al. [4]. (It may be noted that these authors' calculations extend to $x_w/M = 20$ even though their measurements extend to $x_w/M = 85$. As with our calculation, it appears that the agreement between their theory and experiment deteriorates beyond $x_w/M = 10$.)

In order to calculate profiles of $\langle \phi'^2 \rangle$ we need one more piece of information. Assuming $\langle \phi^2 \rangle$ to have a Gaussian profile [4] with variance $X\sigma^2$, then the profiles of $\langle \phi'^2 \rangle$ can be determined from σ, I and X. But from our analysis X is not known. If there were only flapping (i.e. $I_c = 0$), then X could be determined, since $\langle \phi'^2 \rangle_{\text{flap}}$ is known: the value of X obtained thus we denote by X_{flap}. In the early stages of the wake, flapping is the dominant contribution to I, and X_{flap} can be expected to be a good approximation to X. Figure 6 shows normalized rms profiles

$$\phi'/\phi'_0 \equiv (\langle \phi'^2 \rangle / \langle \phi'^2 \rangle_{z=0})^{1/2}, \tag{33}$$

at the first three measuring stations based on the calculated values of σ, I and X_{flap}. It may be seen that there is good agreement with the data. The location and magnitude of the peak away from the plane of symmetry is calculated reasonably well.

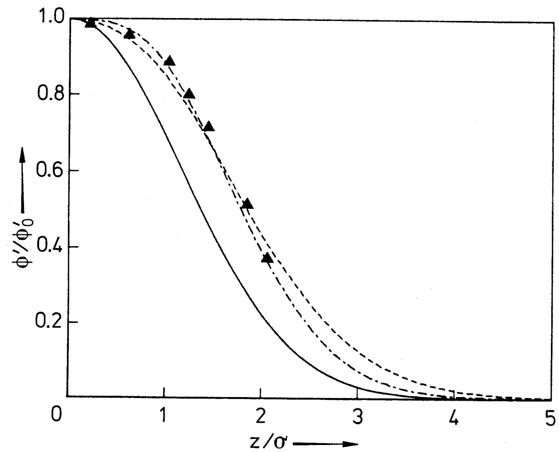

Fig. 7. Normalized standard deviation against cross-stream distance at $x_w/M = 100$: conditional pdf method. —— $C_\phi = 2.0$, $X = X_{\text{flap}} = 1.59$, ---- $C_\phi = 2.0$, $X = 1.00$, —·—·— $C_\phi = 4.0$, $X = 1.39$, ▲ data [3]

At the measurement stations further downstream, the peak value of $\langle \phi'^2 \rangle$ is on the plane of symmetry and the profiles are approximately self similar. Figure 7 shows a comparison of the measured and calculated profiles at the last measurement station $x_w/M = 100$. Here, I_c is the dominant contribution to I, and it would be fortuitous if X_{flap} approximated X. It can be seen that the calculated profile (with $C_\phi = 2.0$) based on σ, I and $X_{\text{flap}} = 1.59$ is significantly narrower than the experimental profile: but with the value $X = 1.0$, the agreement is improved. Better agreement still is obtained with $C_\phi = 4.0$, $X = 1.39$.

Discussion

Both the conditional and unconditional pdf methods are capable of describing the mean spreading of the thermal wake through all three stages of its development. The ability is due to the fact that convection appears in closed form in the pdf equations, and gradient-diffusion models are avoided.

The conditional pdf method succeeds in calculating the temperature variance (to within a factor of 2) whereas the unconditional method does not. The crucial difference between the two methods is revealed in the (unconditional) integral variance equation:

$$\frac{dI}{dt} = \bar{P} - 2\bar{\varepsilon}_\phi, \tag{34}$$

where

$$\bar{\varepsilon}_\phi \equiv \int_{-\infty}^{\infty} \varepsilon_\phi \, dz. \tag{35}$$

The integral production $\bar{P}$ is the same for both models (see Eq. 13). For the unconditional model, the integral dissipation is

$$\bar{\varepsilon}_\phi = 1/2\, C_\phi \frac{\varepsilon}{k} I, \tag{36}$$

whereas for the conditional model it is

$$\bar{\varepsilon}_\phi = 1/2\, C_\phi \frac{\varepsilon}{k}(I - I_{\text{flap}}) + \frac{\Gamma}{4\sqrt{\pi}}(\tilde{\sigma}^{-3} - \sigma^{-3}). \tag{37}$$

Figures 3 and 5 show that the modelled terms (that contains C_ϕ) have no effect until beyond the first measurement station. The success of the conditional model in calculating the variance at $x_w/M = 0.36$ is due, therefore, to the last term in Eq. 37. (At this location, the unconditional model yields a variance larger by a factor of 3.) The term – which is exact (given σ and $\tilde{\sigma}$) – is due to the steep gradients in the conditional mean profile. From a broader perspective, the success of the conditional model is due to its ability to separate the flapping of the wake (largely determined by w_0) from the more random spreading.

Several approximations are made in the Appendix in order to obtain analytic solutions. The errors thus introduced can be avoided by solving the modelled conditional pdf transport equation by a Monte Carlo method.

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Appendix

An analytic solution for the conditional mean $\langle \phi \,|\, \tilde{w} \rangle$ can be obtained from the modelled joint pdf equation (Eqs. 25–28) with three approximations. First, the conditional pressure gradient in Eq. 26 is neglected. This term is small and its neglect is an excellent approximation. Second, the scalar dissipation term (Eq. 27) is omitted. The principal justification here is that the scalar dissipation does not affect the mean directly. Third, the condition $w(\boldsymbol{x} = 0, t = 0) = \tilde{w}$ is replaced by the (unlikely) condition $w(x, y, z = 0, t = 0) = \tilde{w}$ (all x and y). This is identical to the conventional assumption that dispersion can be approximated as a one-dimensional phenomenon.

With these approximations the modelled conditional joint pdf equation becomes

$$\frac{\partial \tilde{f}}{\partial t} + V_i \frac{\partial \tilde{f}}{\partial x_i} = \Gamma \nabla^2 \tilde{f} + \left(\frac{1}{2} + \frac{3}{4} C_0\right) \frac{\varepsilon}{k} \frac{\partial}{\partial V_i} \{\tilde{f}(V_i - \langle U_i \,|\, \tilde{w} \rangle)\} + \frac{1}{2} C_0 \varepsilon \frac{\partial^2 \tilde{f}}{\partial V_i \, \partial V_i}. \tag{A1}$$

This is also the joint pdf equation [13] corresponding to the following random walk of a particle with position $\hat{\boldsymbol{x}}(t)$ and velocity $\hat{\boldsymbol{U}}(t)$

$$\hat{\boldsymbol{x}}(t + \Delta t) = \hat{\boldsymbol{x}}(t) + \hat{\boldsymbol{U}}(t)\,\Delta t + (2\Gamma \Delta t)^{1/2} \boldsymbol{\xi}, \tag{A2}$$

$$\hat{\boldsymbol{U}}(t + \Delta t) = \hat{\boldsymbol{U}}(t)\left[1 - \left(\frac{1}{2} + \frac{3}{4} C_0\right) \frac{\varepsilon}{k} \Delta t\right] + (C_0 \varepsilon \Delta t)^{1/2} \boldsymbol{\xi}', \tag{A3}$$

(in the limit of $\Delta t \to 0$), where $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ are independent joint-normal random vectors with zero mean and unit diagonal covariance matrix. For the case considered, motion in all but the $z = x_3$ direction is irrelevant. The appropriate initial conditions are $\hat{z}(t = 0) = 0$, $\hat{w}(t = 0) = \tilde{w}$.

A standard analysis of the random walk [20] shows that the mean profile of a diffusing substance (the excess temperature in the present case) is represented by the pdf of the position of particles leaving the source (at $z = 0$) at time $t = 0$. Further, it is shown that this pdf is Gaussian [20]. We can therefore write

$$\tilde{\phi} \equiv \langle \phi(z,t) \,|\, \tilde{w} \rangle = \frac{1}{\sqrt{2\pi}\,\tilde{\sigma}} \exp(-(z-\tilde{z})^2/2\tilde{\sigma}^2), \tag{A 4}$$

where $\tilde{z}(\tilde{w},t)$ and $\tilde{\sigma}^2(\tilde{w},t)$ are the conditional mean and variance. These quantities can be expressed in terms of the Lagrangian position $\hat{z}(t)$ and velocity $\hat{w}(t)$ of fluid particles originating from $z = 0$ at time $t = 0$ with velocity $\hat{w}(0) = \tilde{w}$:

$$\tilde{z}(\tilde{w},t) = \langle \hat{z}(t) \,|\, \tilde{w} \rangle = \int_0^t \langle \hat{w}(t') \,|\, \tilde{w} \rangle \, dt', \tag{A 5}$$

and

$$\begin{aligned} \tilde{\sigma}^2(\tilde{w},t) &= \langle \hat{z}(t)^2 \,|\, \tilde{w} \rangle - \tilde{z}(\tilde{w},t)^2 \\ &= 2\int_0^t \int_0^{t''} \langle \hat{w}(t')\, \hat{w}(t'') \,|\, \tilde{w} \rangle \, dt' \, dt'' + 2\Gamma t - \tilde{z}(\tilde{w},t)^2. \end{aligned} \tag{A 6}$$

In the following analysis, $\tilde{z}$ and $\tilde{\sigma}^2$ are determined.

For the decaying turbulent flow under consideration, ε and k are functions of t; $\varepsilon \propto (t_0 + t)^{-(m+1)}$ and $\varepsilon/k = m/(t_0 + t)$ where $t_0 \equiv x_0/U$. From the random walk in Eq. A 3 we then obtain

$$\begin{aligned} \langle \hat{w}(t) \rangle &= \hat{w}(0) \exp\left(-\beta \int_0^t (\varepsilon/k) \, dt \right) \\ &= \hat{w}(0)\,(1 + t/t_0)^{-\beta m}, \end{aligned} \tag{A 7}$$

where

$$\beta = (\tfrac{1}{2} + \tfrac{3}{4} C_0). \tag{A 8}$$

Thus

$$\langle \hat{w}(t) \,|\, \tilde{w} \rangle = \tilde{w}(1 + t/t_0)^{-\beta m}. \tag{A 9}$$

Substitution of Eq. A 9 in Eq. A 5 yields

$$\tilde{z}(\tilde{w},t) = \tilde{w}\, t_0 [1 - (1 + t/t_0)^{-s}]/s, \tag{A 10}$$

where $s \equiv \dfrac{m}{2}(\tfrac{3}{2} C_0 + 1) - 1$.

Now, in order to determine $\tilde{\sigma}^2$, let

$$\tilde{\sigma}^2 = 2\Gamma t + \Delta_0^2, \tag{A 11}$$

where, from Eq. A 6, Δ_0^2 is defined by

$$\Delta_0^2 = 2\int_0^t \int_0^{t''} \langle \hat{w}(t')\, \hat{w}(t'') \,|\, \tilde{w} \rangle \, dt' \, dt'' - \tilde{z}^2. \tag{A 12}$$

We write

$$\langle \hat{w}(t')\, \hat{w}(t'') \,|\, \tilde{w} \rangle = \int_{-\infty}^{\infty} g'(w', t' \,|\, \tilde{w})\, \langle \hat{w}(t')\, \hat{w}(t'') \,|\, \hat{w}(t') = w' \rangle \, dw', \tag{A 13}$$

where $g'(w', t' \,|\, \tilde{w})$ is the pdf of $\hat{w}$ at time t' conditional upon $\hat{w}(0) = \tilde{w}$ which is also Gaussian [21]:

$$g'(w', t' \,|\, \tilde{w}) = \frac{1}{\sqrt{2\pi}\, W} \exp\left(-\,(w' - \langle \hat{w} \,|\, \tilde{w} \rangle)^2 / 2W^2\right), \tag{A 14}$$

where

$$W^2(\tilde{w}, t') = \langle \hat{w}(t')^2 \,|\, \tilde{w} \rangle - \langle \hat{w}(t') \,|\, \tilde{w} \rangle^2. \tag{A 15}$$

$W^2(\tilde{w}, t)$ is the solution of the following differential equation derived from the random walk in Eq. A 3:

$$\partial W^2 / \partial t = C_0 \varepsilon - 2\beta(\varepsilon/k)\, W^2, \tag{A 16}$$

with the initial condition $W^2(\tilde{w}, 0) = 0$. It may be seen from Eq. A 16 that, with the specified initial condition, W^2 is independent of $\tilde{w}$.

Further, the two-time Lagrangian velocity correlation for the random walk in Eq. A 3 can be shown to be

$$\langle \hat{w}(t')\, \hat{w}(t'') \,|\, w(t') = w' \rangle = w'^2 [(t_0 + t')/(t_0 + t'')]^{s+1} \quad \text{for } t' \leqq t''. \tag{A 17}$$

Substitution of Eq. A 17 and the solution of Eq. A 16 into Eq. A 13 and subsequently into Eq. A 12 yields

$$\Delta_0^2 = 2\langle w_0^2 \rangle\, t_0^2 \left\{ \frac{-(r+s)}{2s^2(r-s)} + \frac{(1+t/t_0)^{r-s}}{r(r-s)} + \frac{(r+s)(1+t/t_0)^{-s}}{rs^2} - \frac{(1+t/t_0)^{-2s}}{2s^2} \right\}, \tag{A 18}$$

where $r \equiv 1/2\, m(3/2\, C_0 - 1) + 1$. Thus the conditional profile $\langle \phi \,|\, \tilde{w} \rangle$ is completely determined from Eqs. A 4, A 10, A 11 and A 18.

From the conditional profile and Eq. 22 it can be seen that the unconditional profile is indeed Gaussian and the unconditional wake width σ is determined to be

$$\sigma^2 = \tilde{\sigma}^2 + [\tilde{z}(\langle w_0^2 \rangle^{1/2}, t)]^2 \tag{A 19}$$

$$= 2\Gamma t + \Delta^2 \tag{A 20}$$

where

$$\Delta^2 = \Delta_0^2 + [\tilde{z}(\langle w_0^2 \rangle^{1/2}, t)]^2 = 2\langle w_0^2 \rangle\, t_0^2 \left\{ \frac{(1+t/t_0)^{r-s}}{r(r-s)} + \frac{(1+t/t_0)^{-s}}{rs} - \frac{1}{s(r-s)} \right\}. \tag{A 21}$$

Equation A 20 shows that σ^2 is composed of a contribution $(2\Gamma t)$ due to molecular diffusion, and a contribution (Δ^2) due to turbulent dispersion. The latter contribution scales entirely with turbulent quantities. It is readily deduced from the power law decay Eq. 1 that $(\langle w_0^2 \rangle\, t_0^2)^{1/2}$ is proportional to the length scale $l_0 \equiv k^{3/2}/\varepsilon$ at the wire:

$$\langle w_0^2 \rangle^{1/2}\, t_0 = (2/3)^{1/2}\, m\, l_0. \tag{A 22}$$

Thus Eq. A 8 can be rewritten

$$(\Delta/l_0)^2 = \frac{4m^2}{3}\left\{\frac{(1+x_w/x_0)^{r-s}}{r(r-s)} + \frac{(1+x_w/x_0)^{-s}}{rs} - \frac{1}{s(r-s)}\right\}. \tag{A 23}$$

This equation predicts that, for different conditions (i.e. different U, x_0/M, and power-law constant A), the quantity Δ/l_0 depends only on x_w/x_0 and m. (From flow to flow, m may vary by 15%.)

From the conditional and unconditional mean profiles, the flapping contribution to $\langle\phi'^2\rangle$ can be determined. Performing the integration in Eq. 30 yields

$$\langle\phi'^2\rangle_{\text{flap}} = \frac{1}{2\pi}\left\{\frac{\exp(-\frac{1}{2}z^2/\sigma_*^2)}{\sqrt{2}\,\tilde{\sigma}\,\sigma_*} - \frac{\exp(-z^2/\sigma^2)}{\sigma^2}\right\}, \tag{A 24}$$

where

$$\sigma_*^2 = \tfrac{1}{2}\tilde{\sigma}^2 + [\tilde{z}(\langle w_0^2\rangle^{1/2}, t)]^2. \tag{A 25}$$

And, integrating over z, the integral variance due to flapping is found to be

$$I_{\text{flap}} = [1/\tilde{\sigma} - 1/\sigma]/(2\sqrt{\pi}). \tag{A 26}$$

Noting that the shape of the profile of $\tilde{\phi}$ (though not its position) is independent of $\tilde{w}$ Eq. A 18, we assume, similarly, that the shape of the profile of $\widetilde{\phi''^2}$ is also independent of $\tilde{w}$. Then, the integral variance due to conditional fluctuations is, from Eqs. 31 and 32,

$$I_c = \int \widetilde{\phi''^2}\, dz. \tag{A 27}$$

From the modelled conditional pdf equation, a modelled equation for ϕ''^2 can be obtained. This can be integrated over z to obtain

$$\frac{dI_c}{dt} = \bar{P}_c - C_\phi \frac{\varepsilon}{k} I_c, \tag{A 28}$$

where, from the fact that $\tilde{\phi}$ is Gaussian,

$$\bar{P}_c \equiv \left(\frac{1}{2}\frac{d\tilde{\sigma}^2}{dt} - \Gamma\right)\bigg/(2\tilde{\sigma}^3\sqrt{\pi}). \tag{A 29}$$

(It may be noted that these equations are the same (mutadis mutandis) as Eqs. 12 and 13 for the unconditional model.)

The solution described above applies to a line source. In the experiment, the heated wire is a source of finite size which Stapountzis et al. [4] suggest modelling as a Gaussian temperature profile of size $\sigma_0 = 1.25\,d$. Such an initial profile can be accommodated in the analysis simply by changing $2\Gamma t$ to $\sigma_0^2 + 2\Gamma t$ in Eqs. A 11 and A 20. The reported variance calculations were performed with $\sigma_0 = 1.25\,d$, but this produced only a small change in I compared to calculations with $\sigma_0 = 0$.

References

1. Townsend, A. A. (1954): The diffusion behind a line source in homogeneous turbulence. Proc. Roy. Soc. **A 224**, 487
2. Uberoi, M. S., Corrsin, S. (1953): Diffusion of heat from a line source in isotropic turbulence. NACA Report 1142
3. Warhaft, Z. (1984): Interference of line sources in grid turbulence. J. Fluid Mech. **44,** 363
4. Stapountzis, H., Sawford, B. L., Hunt, J. C. R., Britter, R. E. (1983): Structure of the temperature field downwind of a line source in grid turbulence. Submitted to J. Fluid Mech.
5. Taylor, G. I. (1921): Diffusion by continuous movements. Proc. Lond. Math. Soc. **20**, 196
6. Gifford, F. (1959): Statistical properties of a fluctuating plume dispersal model. Adv. Geophys. **6**, 117
7. Sykes, R. I., Lewellen, W. S., Parker, S. F. (1984): A turbulent transport model for concentration fluctuations and fluxes. J. Fluid Mech. **139,** 193
8. Sawford, B. L., Hunt, J. C. R. (1983): Effects of turbulence structure, molecular diffusion and source size on fluctuations of concentration in homogeneous turbulence. Submitted to J. Fluid Mech.
9. Deardorff, J. W. (1978): Closure of second- and third-order rate equations for diffusion in homogeneous turbulence. Phys. Fluids **21**, 525
10. Libby, P. A., Scragg, C. A. (1973): Diffusion of heat from a line source downstream of a turbulence grid. AIAA J. **11**, 562
11. Fischer, H. B., List, E. J., Koh, R. C. Y., Imberger, J., Brooks, N. H.: *Mixing in Inland and Coastal Waters*, (Academic, New York 1979)
12. Pope, S. B. (1981): Transport equation for the joint probability density function of velocity and scalars in turbulent flow. Phys. Fluids **24**, 588
13. Pope, S. B. (1985): PDF methods for turbulent reactive flows. To be published in Prog. Energy Combust. Sci.
14. Launder, B. E., Spalding, D. B.: *Mathematical Models of Turbulence*, (Academic, New York 1972)
15. Lumley, J. L. (1978): Computational modelling of turbulent flows. Adv. Appl. Mech. **18**, 123
16. Pope, S. B. (1983): Consistent modelling of scalars in turbulent flows. Phys. Fluids **26**, 404
17. Warhaft, Z., Lumley, J. L. (1978): An experimental study of the decay of temperature fluctuations in grid generated turbulence. J. Fluid Mech. **99**, 545
18. Tavoularis, S., Corrsin, S. (1981): Experiments in nearly homogeneous turbulent shear flow with a uniform mean temperature gradient. Part I. J. Fluid Mech. **104**, 311
19. Monin, A. S., Yaglom, A. M.: *Statistical Fluid Mechanics*, Vol. 2, pp. 358–361 (MIT Press, Cambridge 1975)
20. Chandrasekhar, S.: In *Noise and Stochastic Processes*, ed. by N. Wax (Dover, New York 1954)
21. Uhlenbeck, G. E., Ornstein, L. S.: In *Noise and Stochastic Processes*, ed. by N. Wax (Dover, New York 1954)
22. Pope, S. B. (1983): A Lagrangian two-time probability density function equation for inhomogeneous turbulent flows. Phys. Fluids **26**, 3448