Turbulent Flows
Stephen B. Pope
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Solution to Exercise 3.9

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If $Q(y)$ is a monotonic function, then its inverse exists. Using the definition of the CDF, as well as the fact that if $Q(y)$ is an increasing function, so is $Q^{-1}(y)$, we obtain

$$F_Y(y) = P(Y < y) = P(Q(U) < y) = P(U < Q^{-1}(y)). \tag{1}$$

Keeping in mind that $Q^{-1}(y) = V$ we get:

$$F_Y(y) = F(Q^{-1}(y)) = F(V). \tag{2}$$

The definition of the PDF gives

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{dF(Q^{-1}(y))}{dy}. \tag{3}$$

Using the chain rule and noting that

$$\frac{dQ^{-1}(y)}{dy} = \left(\frac{dQ(V)}{dV}\right)^{-1} \tag{4}$$

we obtain

$$f_Y(y) = f(V) / \left(\frac{dQ(V)}{dV}\right). \tag{5}$$

If $Q(y)$ is a decreasing function on the other hand we get:

$$F_Y(y) = P(Q(U) < y) = P(U > Q^{-1}(y)) = 1 - P(U < Q^{-1}(y)) \tag{6}$$

which gives

$$F_Y(y) = 1 - F(Q^{-1}(y)) = 1 - F(V). \tag{7}$$

Using the same procedure to obtain the PDF, the final result is:

$$f_Y(y) = -f(V) / \left(\frac{dQ(V)}{dV}\right). \tag{8}$$

Equations 5 and 8 can be written (using the appropriate signs)

$$f_Y(y)\left(\pm\frac{dQ(V)}{dV}\right) = f(V). \tag{9}$$

Using the fact that if $Q(y)$ is increasing/decreasing the derivative is positive/negative, and multiplying by dV on both sides we get

$$f_Y(y)\left|\frac{dQ(V)}{dV}\right| dV = f(V)dV, \tag{10}$$

which, by using the definition of dy gives us

$$f_Y(y)dy = f(V)dV. \tag{11}$$