Turbulent Flows
Stephen B. Pope
Cambridge University Press (2000)

Solution to Exercise 11.4

Prepared by: Zhuyin Ren *Date:* 04/17/03

According to Eq.(11.28),

$$6\eta^2 = b_{ij}b_{ji} = \delta_{ik}\Pi_{ik}, \tag{1}$$

where

$$\Pi = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & (\lambda_1+\lambda_2)^2 \end{bmatrix}. \tag{2}$$

So

$$\eta^2 = \frac{1}{3}(\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2). \tag{3}$$

According to Eq.(11.29),

$$6\xi^3 = b_{ij}b_{jk}b_{ki} = \delta_{im}\Xi_{im}, \tag{4}$$

where

$$\Xi = \begin{bmatrix} \lambda_1^3 & 0 & 0 \\ 0 & \lambda_2^3 & 0 \\ 0 & 0 & -(\lambda_1+\lambda_2)^3 \end{bmatrix}. \tag{5}$$

So

$$\xi^3 = \frac{1}{6}(\lambda_1^3 + \lambda_2^3 - (\lambda_1+\lambda_2)^3) = -\frac{1}{2}\lambda_1\lambda_2(\lambda_1+\lambda_2). \tag{6}$$

The eigenvalues of **b** (λ_1,λ_2) are related to $\langle\tilde{u}_1^2\rangle$, $\langle\tilde{u}_2^2\rangle$ and $\langle\tilde{u}_3^2\rangle$ through

$$\lambda_1 = \frac{\langle\tilde{u}_1^2\rangle}{\langle\tilde{u}_1^2\rangle + \langle\tilde{u}_2^2\rangle + \langle\tilde{u}_3^2\rangle} - \frac{1}{3}, \tag{7}$$

and

$$\lambda_2 = \frac{\langle\tilde{u}_2^2\rangle}{\langle\tilde{u}_1^2\rangle + \langle\tilde{u}_2^2\rangle + \langle\tilde{u}_3^2\rangle} - \frac{1}{3}. \tag{8}$$

a:) isotropic; $\langle\tilde{u}_1^2\rangle$=$\langle\tilde{u}_2^2\rangle$=$\langle\tilde{u}_3^2\rangle$;
For this case,

$$\lambda_1 = \lambda_2 = \lambda_3 = 0, \tag{9}$$

so

$$\eta = 0, \qquad \xi = 0. \tag{10}$$

b:) two-component, axisymmetric:$\langle\tilde{u}_1^2\rangle$=$\langle\tilde{u}_2^2\rangle$, $\langle\tilde{u}_3^2\rangle$=0;
For this case,

$$\lambda_1 = \frac{\langle\tilde{u}_1^2\rangle}{\langle\tilde{u}_1^2\rangle + \langle\tilde{u}_2^2\rangle + \langle\tilde{u}_3^2\rangle} - \frac{1}{3} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}, \tag{11}$$

and

$$\lambda_2 = \frac{\langle\tilde{u}_2^2\rangle}{\langle\tilde{u}_1^2\rangle + \langle\tilde{u}_2^2\rangle + \langle\tilde{u}_3^2\rangle} - \frac{1}{3} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}, \tag{12}$$

so

$$\eta = \sqrt{\frac{1}{3}(\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2)} = \frac{1}{6}, \tag{13}$$

and

$$\xi = \left(-\frac{1}{2}\lambda_1\lambda_2(\lambda_1 + \lambda_2)\right)^{1/3} = -\frac{1}{6}. \tag{14}$$

c:) one-component; $\langle\tilde{u}_2^2\rangle$=$\langle\tilde{u}_3^2\rangle$=0;
For this case

$$\lambda_1 = \frac{\langle\tilde{u}_1^2\rangle}{\langle\tilde{u}_1^2\rangle + \langle\tilde{u}_2^2\rangle + \langle\tilde{u}_3^2\rangle} - \frac{1}{3} = 1 - \frac{1}{3} = \frac{2}{3}, \tag{15}$$

and

$$\lambda_2 = \frac{\langle\tilde{u}_2^2\rangle}{\langle\tilde{u}_1^2\rangle + \langle\tilde{u}_2^2\rangle + \langle\tilde{u}_3^2\rangle} - \frac{1}{3} = 0 - \frac{1}{3} = -\frac{1}{3}, \tag{16}$$

so

$$\eta = \sqrt{\frac{1}{3}(\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2)} = \frac{1}{3}, \tag{17}$$

and

$$\xi = \left(-\frac{1}{2}\lambda_1\lambda_2(\lambda_1 + \lambda_2)\right)^{1/3} = \frac{1}{3}. \tag{18}$$

d:) axisymmetric Reynolds stresses (i.e., $\langle \tilde{u}_1^2 \rangle = \langle \tilde{u}_2^2 \rangle$, $\lambda_1 = \lambda_2$);
For this case,

$$\eta^2 = \frac{1}{3}(\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2) = \lambda_1^2, \tag{19}$$

and

$$\xi^3 = -\frac{1}{2}\lambda_1\lambda_2(\lambda_1 + \lambda_2) = -\lambda_1^3. \tag{20}$$

If there is one large eigenvalues (i.e., $\langle \tilde{u}_1^2 \rangle = \langle \tilde{u}_2^2 \rangle < \langle \tilde{u}_3^2 \rangle$), then by Eq. 7 and Eq. 8, it is easy to show that λ_1 and λ_2 are negative and $-\frac{1}{3} < \lambda_1 = \lambda_2 < 0$. So

$$\xi = -\lambda_1, \quad \eta = |\lambda_1| = -\lambda_1, \tag{21}$$

i.e.

$$\xi = \eta = -\lambda, \quad \xi > 0. \tag{22}$$

If there is one small eigenvalues (i.e., $\langle \tilde{u}_1^2 \rangle = \langle \tilde{u}_2^2 \rangle > \langle \tilde{u}_3^2 \rangle$), then by Eq. 7 and Eq. 8, it is easy to show that λ_1 and λ_2 are positive and $0 < \lambda_1 = \lambda_2 < \frac{1}{6}$. So

$$\xi = -\lambda_1, \quad \eta = \lambda_1, \tag{23}$$

i.e.

$$\xi = -\eta, \quad \xi < 0. \tag{24}$$

Given the normal stresses in principal axes,

$$\begin{aligned} F &= \frac{27\langle \tilde{u}_1^2 \rangle \langle \tilde{u}_2^2 \rangle \langle \tilde{u}_3^2 \rangle}{(\langle \tilde{u}_1^2 \rangle + \langle \tilde{u}_2^2 \rangle + \langle \tilde{u}_3^2 \rangle)^3} \\ &= (1 + 3\lambda_1)(1 + 3\lambda_2)(1 - 3\lambda_1 - 3\lambda_2) \\ &= 1 - 9(\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2) - 27\lambda_1\lambda_2(\lambda_1 + \lambda_2) \\ &= 1 - 27\eta^2 + 54\xi^3 \\ &= 1 - \frac{9}{2}b_{ii}^2 + 9b_{ii}^3, \end{aligned} \tag{25}$$

where the first step follows the definition of F and the third step follows Eq. 7 and Eq. 8. In two-component turbulence, $F = 0$. So according to Eq. 25,

$$1 - 27\eta^2 + 54\xi^3 = 0, \tag{26}$$

i.e.

$$\eta = \left(\frac{1}{27} + 2\xi^3 \right)^{1/2} . \tag{27}$$