

**Turbulent Flows**  
Stephen B. Pope  
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**Solution to Exercise 10.8**

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From the definition of the turbulence frequency  $\omega \equiv \varepsilon/k$ , we obtain after isolating for  $\varepsilon$  and by using the product rule

$$\frac{d\varepsilon}{dt} = \frac{d\omega}{dt}k + \omega \frac{dk}{dt}. \quad (1)$$

From Eq.(10.64) with the chain rule applied on  $\omega^2$  and isolated for  $d\omega/dt$  we get

$$\frac{d\omega}{dt} = -\frac{\alpha}{2} \left( \omega^2 + \frac{\mathcal{S}^2}{\beta^2} \right). \quad (2)$$

Substituting  $d\omega/dt$  and  $dk/dt$  in Eq.(1) with Eq.(2) and Eq.(10.55), respectively, leads to

$$\frac{d\varepsilon}{dt} = -\frac{\alpha k}{2} \left( \omega^2 + \frac{\mathcal{S}^2}{\beta^2} \right) + \omega(\mathcal{P} - \varepsilon). \quad (3)$$

Again using the definition of the turbulence frequency,  $\omega$  is eliminated from Eq.(3) by

$$\begin{aligned} \frac{d\varepsilon}{dt} &= -\frac{\alpha k}{2} \left[ \left( \frac{\varepsilon}{k} \right)^2 + \frac{\mathcal{S}^2}{\beta^2} \right] + \frac{\varepsilon}{k}(\mathcal{P} - \varepsilon) \\ &= \frac{\mathcal{P}\varepsilon}{k} + \frac{\alpha k \mathcal{S}^2}{2\beta^2} - \left( 1 + \frac{\alpha}{2} \right) \frac{\varepsilon^2}{k}, \end{aligned} \quad (4)$$

which is equivalent to Eq.(10.67).

Using the simplified expression for  $\mathcal{P}/\varepsilon$  (Eq.(10.50)), derived in exercise 10.5 for homogeneous shear flow,  $\mathcal{S}^2$  can be isolated from Eq.(10.50) and eliminated in Eq.(4) leading to

$$\begin{aligned} \frac{d\varepsilon}{dt} &= \frac{\mathcal{P}\varepsilon}{k} + \frac{\alpha k \mathcal{P}\varepsilon}{2\beta^2 C_\mu k^2} - \left( 1 + \frac{\alpha}{2} \right) \frac{\varepsilon^2}{k} \\ &= \left( 1 + \frac{\alpha}{2\beta^2 C_\mu} \right) \frac{\mathcal{P}\varepsilon}{k} - \left( 1 + \frac{\alpha}{2} \right) \frac{\varepsilon^2}{k}, \end{aligned} \quad (5)$$

which is equivalent to Eq.(10.68). Comparing Eq.(5) and Eq.(10.56), the relations

$$C_{\varepsilon 1} = 1 + \frac{\alpha}{2\beta^2 C_\mu} \quad \text{and} \quad C_{\varepsilon 2} = 1 + \frac{\alpha}{2} \quad (6)$$

are resulting. Isolating this system for  $\alpha$  and  $\beta$ , leads to

$$\alpha = 2(C_{\varepsilon 2} - 1) \quad \text{and} \quad \beta = \sqrt{\frac{C_{\varepsilon 2} - 1}{C_\mu(C_{\varepsilon 1} - 1)}}, \quad (7)$$

which is equivalent to Eq.(10.65) and Eq.(10.66).

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