Turbulent Flows
Stephen B. Pope
Cambridge University Press (2000)

Solution to Exercise 10.8

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From the definition of the turbulence frequency $\omega \equiv \varepsilon/k$, we obtain after isolating for ε and by using the product rule

$$\frac{d\varepsilon}{dt} = \frac{d\omega}{dt}k + \omega\frac{dk}{dt}. \tag{1}$$

From Eq.(10.64) with the chain rule applied on ω^2 and isolated for $d\omega/dt$ we get

$$\frac{d\omega}{dt} = -\frac{\alpha}{2}\left(\omega^2 + \frac{\mathcal{S}^2}{\beta^2}\right). \tag{2}$$

Substituting $d\omega/dt$ and dk/dt in Eq.(1) with Eq.(2) and Eq.(10.55), respectively, leads to

$$\frac{d\varepsilon}{dt} = -\frac{\alpha k}{2}\left(\omega^2 + \frac{\mathcal{S}^2}{\beta^2}\right) + \omega(\mathcal{P} - \varepsilon). \tag{3}$$

Again using the definition of the turbulence frequency, ω is eliminated from Eq.(3) by

$$\begin{aligned}\frac{d\varepsilon}{dt} &= -\frac{\alpha k}{2}\left[\left(\frac{\varepsilon}{k}\right)^2 + \frac{\mathcal{S}^2}{\beta^2}\right] + \frac{\varepsilon}{k}(\mathcal{P} - \varepsilon) \\ &= \frac{\mathcal{P}\varepsilon}{k} + \frac{\alpha k \mathcal{S}^2}{2\beta^2} - \left(1 + \frac{\alpha}{2}\right)\frac{\varepsilon^2}{k},\end{aligned} \tag{4}$$

which is equivalent to Eq.(10.67).

Using the simplified expression for $\mathcal{P}/\varepsilon$ (Eq.(10.50)), derived in exercise 10.5 for homogeneous shear flow, $\mathcal{S}^2$ can be isolated from Eq.(10.50) and eliminated in Eq.(4) leading to

$$\begin{aligned}\frac{d\varepsilon}{dt} &= \frac{\mathcal{P}\varepsilon}{k} + \frac{\alpha k \mathcal{P}\varepsilon}{2\beta^2 C_\mu k^2} - \left(1 + \frac{\alpha}{2}\right)\frac{\varepsilon^2}{k} \\ &= \left(1 + \frac{\alpha}{2\beta^2 C_\mu}\right)\frac{\mathcal{P}\varepsilon}{k} - \left(1 + \frac{\alpha}{2}\right)\frac{\varepsilon^2}{k},\end{aligned} \tag{5}$$

which is equivalent to Eq.(10.68). Comparing Eq.(5) and Eq.(10.56), the relations

$$C_{\varepsilon 1} = 1 + \frac{\alpha}{2\beta^2 C_\mu} \quad \text{and} \quad C_{\varepsilon 2} = 1 + \frac{\alpha}{2} \tag{6}$$

are resulting. Isolating this system for α and β, leads to

$$\alpha = 2(C_{\varepsilon 2} - 1) \quad \text{and} \quad \beta = \sqrt{\frac{C_{\varepsilon 2} - 1}{C_\mu (C_{\varepsilon 1} - 1)}}, \tag{7}$$

which is equivalent to Eq.(10.65) and Eq.(10.66).